A NON-PROBABILISTIC APPROACH TO DECISION-MAKING IN THE PRESENCE OF IMPRECISE INFORMATION

By

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The modeling of decision situations often requires the processing of imprecise information into components that describe the decision problems. This dissertation presents quantitative models that accommodate the imprecision resulting from the vagueness and the subjectivity in the assessment of the decision situations. The tools of quantification utilized to represent this imprecision are fuzzy set theory and operation.

Four decision situations are considered. System representations of the decision models are adopted.

The first model is of a single-stage decision process. Both the decision environment and the criterion are assumed to be imprecisely known to the decision maker. The problem is that of choosing among the available decision alternatives. Methods for choosing decisions are discussed. These methods represent the "best" decision, in itself, as a vague concept in accordance to the problem definition.

The second model is that of a multiple-stage decision process and is treated as an extension of the previous model. Methods for composing the various stages of the process and for choosing decision sequences are described. These methods accommodate imprecision in the determination of the "best" decision sequence.

The third model is of a two-person decision process. Conflict between the decision makers is considered. The problem is that of choosing an action (strategy) for each decision maker. New concepts of equilibria and solutions, that accommodate imprecision, are defined.

Finally, a model for a multiple-level decision process is developed. This corresponds to a hierarchy of decision-makers represented by the levels. At each level the decision maker(s) consider the subsequent actions of the decision makers at succeeding levels. Inter-level conflict is represented and imprecision in the problem definition is accommodated.

CHAPTER 1
INTRODUCTION

The modeling of decision problems, in general, involves the determination of two basic components. The first component describes the feasibility of the problem: that is, it defines the collection of possible decisions or actions that may be executed. The second component describes the objectives or decision criteria. This provides a measure against which the feasible decisions can be compared.

The determination of these components involves the processing of data available to the decision maker into information relevant to the problem. In many complex decision situations the data available may not be sufficient to define the problem in an exact and objective form. The processing of such data involves approximating and subjectively assessing the information necessary to describe the decision situation. The ability to capture the vague nature of the information may be the key to examining the concept of a solution to an ill-defined problem.

The tools of probability theory and statistics have been predominantly used in modeling imprecision within decision processes [40]. For example, an approximation is often represented by a value with an associated random error according to a predefined distribution [72]. Also, subjective probabilities are used to model non-objective assessments of information according to an axiomatic representation of human behavior [81]. Determining the error distribution may not be trivial. Also, the decision maker may not behave in a "rational" manner described by the axioms that in turn provide the background for determining subjective probabilities [56]. It is therefore necessary to consider an alternative approach for modeling imprecision such as that arising from approximations and subjectivity.

This dissertation aims at including non-probabilistic representations of approximations and subjective assessments in the modeling of a decision process. The tools used for this purpose are those of fuzzy set theory, first introduced by Zadeh [102]. It also aims at describing how to propagate these representations through the procedure that is said to "solve" the decision problem. Finally it aims at examining concepts of solutions in accordance with the nature of the information that describes the problem.

1.1 Decision Processes

Thus far the term decision process has been used in its most general sense. Here, a systems representation of decision problems is adopted. The tools employed in the modeling of a decision process throughout this dissertation constitute a major departure from traditional approaches. The foundations necessary to establish fuzzy set theory as an alternative tool for modeling imprecise decision processes are yet to be developed. This dissertation includes the development of basic concepts of decision making within the framework of fuzzy set theory.

A simple decision process is first described and is then used as a building block for more complex situations. The following four chapters will each deal with the modeling of a different type of decision situation in increasing order of complexity. They will also deal with incorporating imprecision within these models. The four decision situations are 1) single-stage decision problem; 2) multiple-stage decision problem; 3) two-person decision problem; and 4) multiple-level decision problem. In sequel, these problems are outlined.

1.1.1 Single-Stage Decision Problem

A systems representation of a simple single-stage decision problem is given in Figure 1.1. The information necessary to define the problem can be grouped into three categories. The first category contains information describing the present state of the system. For a well-defined problem the state of the system can be represented as a point in a predefined state space. For example in a production system, the cost and quality of a product may describe

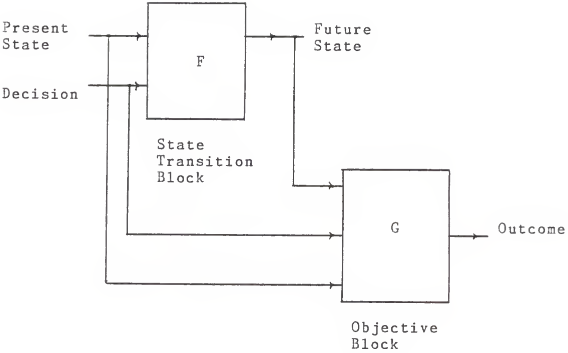


Figure 1.1 Single-stage decision process

the state of the system, yielding a two-dimensional state space of cost and quality. A decision rendered, for example to cut cost, causes the system to undergo a state transition. The second category includes information about the behavior of the system which dictates this transition. This is represented in Figure 1.1 by the block F. The outcome of a decision, in general, depends on the state transition as well as on the decision itself. This dependency is represented by the block G in the diagram. The information about this block is included in the third category and completes the information necessary to define the problem. The decision problem is then that of ranking the outcomes of all possible decisions according to a predefined criterion.

In decision-making situations, determining some or all the information in the categories above may be a source of imprecision in the problem definition. For example, the cost of a product may only be known to the decision maker as an approximation. Similarly, assessing the quality of a product is subjective even when a measure of that quality is predefined. This is because the transition from, say, good to bad quality is gradual rather than abrupt. Also the information about the transition (F) and objective (G) blocks may not be accurately or objectively determined.

In Chapter 2 the single-stage decision problem is modeled using fuzzy set theory to accommodate such imprecisions. The outcomes of decisions are shown to reflect the imprecision in the model. Finally, the ranking of outcomes according to a predefined criterion is discussed.

1.1.2 Multiple-Stage Decision Problem

Figure 1.2 shows the representation of a multiple-stage decision problem. Here it is assumed that a target system state is specified. A feasible sequence of decisions is defined as that which transform the system from its present state to its target state. In Chapter 3, imprecisions in the problem definition are incorporated by a fuzzy set-theoretic representation. Vagueness in the specification of the target state is also included.

The choice of a decision sequence among the feasible ones is determined through a criterion defined on the overall outcome. The question of composing the imprecise intermediate outcome of each stage is addressed in detail. This composition is represented by the block K in the diagram. The outcome of the decision at the first stage is composed with that of the second stage, to form the compounded outcome at the end of the second stage. Subsequent compositions are then performed at every stage. The composition at the final stage determines the overall outcome of a given sequence of decisions. The outcomes of all feasible sequences of decisions can then be ranked using the methods developed in Chapter 2. Chapter 3, in addition, includes an efficient search method for the "best" decision

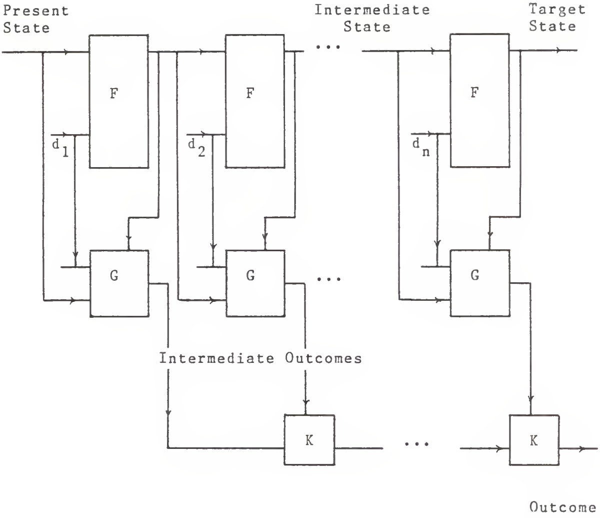


Figure 1.2 Multiple-stage decision process

sequence. This method avoids complete enumeration of all feasible outcomes.

1.1.3 Two-Person Decision Problem

In Chapter 4, a fuzzy set-theoretic representation of imprecision in the normal form of a two-person decision process is introduced. The set of strategies of each decision maker is predefined. A pair of strategies form the decision (or a sequence of decisions) that is rendered to the system. The outcome has two components, one corresponding to each decision maker. The situation is treated as that of conflict, in that the component of the outcome corresponding to each decision maker depends not only on his strategy but also on his opponent's strategy. This does not necessarily constitute a direct conflict (zero-sum game) [56].

Concepts such as equilibrium and solution are defined in the context of the propagated vagueness in the outcome. A representation of a two-person decision process is given in Figure 1.3.

1.1.4 Multiple-Level Decision Problem

In Chapter 5, a multiple-level decision process is considered within a hierarchical systems representation. The treatment of this problem follows the basic principal/agent idea [78]. Figure 1.4.a shows a two-level decision problem, with two decision makers at the second level. The outcome of the decision at level 1 depends on the state transition, as well as the decision itself. The state transition, in the case of a two-level decision system, is determined by three factors: 1) the decision at level one; 2) the decision at level two; and 3) the system behavior as a response to the decisions made.

In analyzing this situation, it will be assumed that the decision-making process of level 2 can be combined with the system behavior and be treated as the overall system observed at level 1. This is depicted in Figure 1.4.b. The state transition block at level 1 (F) includes the decision process of level 2. The level-2 decision process, in general, can itself be a single-stage decision problem (Chapter 2), a multiple-stage decision problem (Chapter 3), or as in the diagram a two-person decision problem (Chapter 4). The hierarchy can be extended to include more than two levels by recursively modeling the level-2 decision process as a multiple-level problem.

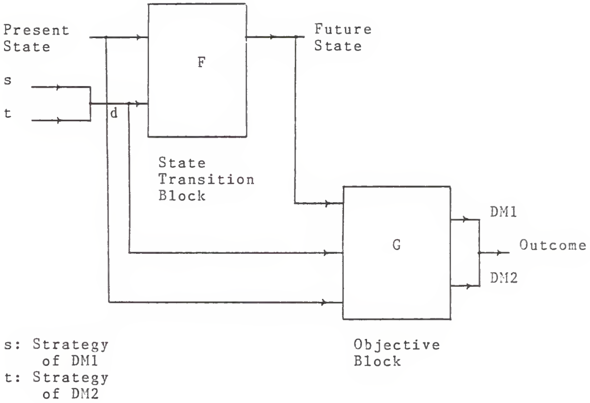
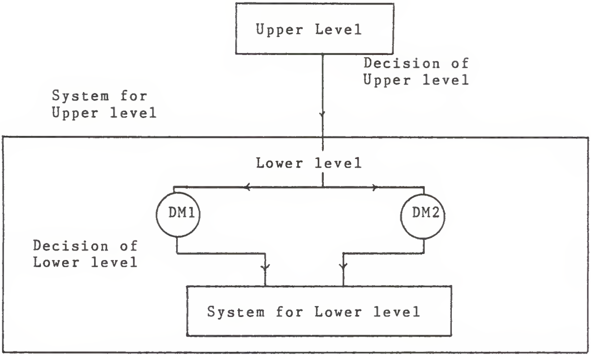
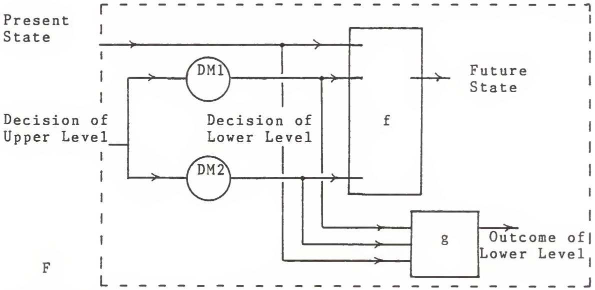


Figure 1.3 Two-person decision process



a) Two-level hierarchy



F: State Transition Block of Upper Level
f: State Transition Block of Lower Level
g: Objective Block of Lower Level

b) System decomposition in a two-level hierarchy

Figure 1.4 Two-level decision process

1.2 Interpretations of Fuzzy Set-Theoretic Representation

The tools chosen to model the vagueness that exists in the outlined decision processes are those of fuzzy set theory [32]. Basic notations, definitions and operations are included in the Appendix.

The attention here is directed toward understanding, motivating and justifying the use of the theory to model imprecision. It is also directed toward interpreting the results of its use in decision modeling. It is for this purpose that this dissertation deals with classes of decision problems rather than with specific situations. The remainder of this chapter is dedicated to demonstrating the use of fuzzy set theory for the purpose at hand. In these demonstrations a clear distinction between the probabilistic and the fuzzy set-theoretic representations is presented. Two special sources of imprecision in the definition of decision problems are considered: namely, approximations and subjective assessments.

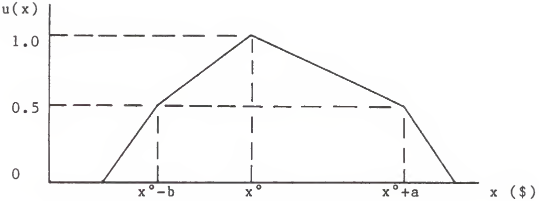
1.2.1 Approximations

In order to represent approximations using the probabilistic approaches, two considerations are imposed. The first is the existence of a "machine" that measures the value that is approximated. The second, and more restricting consideration, is that the error in the measurement of this machine behaves as a random variable with a known probability distribution. When these considerations are met, probability theory provides a rigorous framework within which approximations can be modeled.

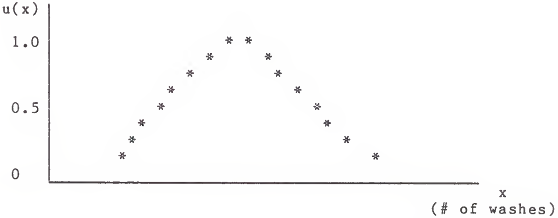
Modeling approximations outside the probabilistic framework may be necessary in decision situations in which approximations are used in their everyday vague sense.

Fuzzy set-theoretic representation is suggested as an alternative to modeling approximations. An interpretation of this representation is given with the help of an exposition. Consider a decision maker approximating the cost of a product by \$x°. This can be represented by the fuzzy set in Figure 1.5.a. The universe is positive dollar amount X whose elements are x in X. The fuzzy set "approximately \$x°" is defined by its membership function u(x). Here "approximately \$x°" is a concept. The value x° represents this concept perfectly. It is therefore associated with a membership value of 1. The values x°+a and x°-b represent the same concept less perfectly and are therefore assigned membership values less than 1. Both values, however, represent that concept equally well. Their membership values are equal, in this case 0.5.

Another interpretation of the membership function can be realized by again adopting the idea that a "machine" measures this approximated value. Here, however, the behavior of this machine is not known. What is known is



a) Fuzzy set of "approximately $x°"



b) Fuzzy set of "good quality article of clothing"

Figure 1.5 Continuous and discrete fuzzy sets

that the machine can operate at different levels of "accuracy." These levels are assigned values between 0 and 1, with 1 being the most accurate. The distribution of error at each accuracy level is not represented. Only the range of its reading for a certain accuracy level is defined. That is, if this machine is most accurate, its reading of the cost of the product is $\$x^{\circ}$. If the machine operates at a less accurate level, say 0.5, then the reading is somewhere between $x^{\circ}+a$ and $x^{\circ}-b$. This representation does not suggest that the level of accuracy at which the machine will operate is known.

1.2.2 Subjective Assessments

A similar argument can be made for the fuzzy set-theoretic representation of subjective assesments. It was earlier suggested that such an assessment, for example, may be required in determining the quality of a product. For the sake of exposition, consider an article of clothing being produced. Also assume that the measure of clothing quality depends on the number of times it can be washed before it tears. If an assessment was made that the article is of "good" quality, then the question is how this assessment can be represented in terms of the number of times the article can be washed.

One approach is to define a subjective probability distribution over the number of washes. In other words, it is to determine the probability that a good quality article of clothing will endure at least a certain number of washes. The determination of such probabilities has been the subject of much research [81]. A hypothetical betting situation is imposed on the decision maker who is in turn assumed to behave in a rational manner [56]. The probabilities are then determined by equating the certainty equivalance to the expected value of the bet. Several variations on this very general idea have been discussed in the literature, some of which relax one or more of the axioms that define rational behavior [83].

An alternative representation of subjective assessments uses fuzzy set theory. Figure 1.5.b shows "good" quality as a fuzzy set. The universe is the positive integers (or the number of washes). "Good" quality is again considered to be a concept, where the membership function is interpreted as above. The membership function is not necessarily increasing. In Figure 1.5.b, it decreases as the number of washes exceed a certain value. This depicts the idea that a large number of washes may represent a concept beyond that of "good" quality, such as "very good" quality. This representation allows the gradual transition of the properties of one subjective assessment into another.

Fuzzy set-theoretic representation of imprecision is used throughout this dissertation. This is done within the context of modeling decision situations. No attempt is made to dictate or describe the actual behavior of decision makers. This acknowleges that a variety of reasonings can be adopted for decision making in the presence of imprecision. Instead, emphasis is placed on the notion that the "best" decision choice, in this context, is in itself a vague concept.

Hereafter, it will be assumed that imprecise information is given as fuzzy sets on predefined universes. Guidelines by which the membership functions can be determined will not be included. This subject was discussed in some detail by Zadeh [105]. The use of some fuzzy set-theoretic operations in certain contexts will be motivated throughout this dissertation.

1.3 Literature Review

Fuzzy set theory has been explored within the fields of decision sciences and operations research. References to previous work are given in Table 1.1. Most of these contributions maintained close ties to traditional approaches while adding the element of "fuzziness" in the problem definition. The need to establish the theory as an alternative tool for modeling some types of imprecision is as yet unfilled.

Table 1.1 Applications of fuzzy set theory in decision sciences and operations research

SUBJECT	REFERENCE NUMBER
Decision Making	
Games	4, 10, 15, 17, 20, 21, 22, 25, 33, 66, 76
General	2, 3, 8, 19, 23, 27, 36, 41, 42, 45, 46, 48, 50, 54, 59, 60, 68, 88, 90, 92, 95, 96, 101, 103, 104, 105, 106, 107, 108
Group Actions	9, 11, 52, 64, 65, 66, 67, 86, 87, 93
Medical Systems	53, 84, 85
Multi-Objective	5, 6, 12, 13, 16, 29, 30, 33, 34, 38, 47, 49, 55, 57, 58, 62, 75, 79, 91, 97, 98, 109
Ranking of Alternatives	1, 7, 15, 18, 32, 35, 69, 82, 93, 94, 99, 100
Operations Research	
Allocation	24, 37
Assignment	51
Dynamic Programming	37, 44
Critical Path and PERT	26
Integer Programming	47
Linear Programming	33, 58, 61, 70, 109
Networks	43, 47
Production Systems	89
Queing	73
Scheduling	74, 77
Transportation	28, 39

To date, fuzzy set-theoretic approaches to decision making have concentrated on developing methods by which vagueness and imprecision can be transformed and interpreted in a well defined and precise manner.

In contrast, this dissertation capitalizes on the idea that in the presence of imprecise information, the choice of decision is in itself an ambiguous concept. The tools of fuzzy set theory are used to consistently represent this idea in various decision models.

Four decision situations are explored. The first is the basic problem of choosing a decision among the available alternatives. Previous work in this field is listed under the heading of ranking of alternatives in Table 1.1. Most of these describe methods by which a real-valued index can be assigned to each fuzzy set representing the outcome of a decision. This index is in turn used to rank these outcomes. The analysis in Chapter 2 also deals with the question of choosing among fuzzy alternatives. This is done using a completely new approach. No attempt is made to conclusively rank the alternatives. The discussion concentrates on determining the "best" choice in a manner that reflects the vagueness in the problem definition.

The second situation explored in this dissertation is the dynamic decision problem. A dynamic problem with fuzzy information was first formulated by Esogbue [37]. The model is an application of resource allocation in cancer research.

In his analysis, Esogbue treats the membership function as an entity to be optimized a long with the associated outcome. The developed dynamic programming algorithm optimizes the membership value at every stage. It then determines the optimal resource allocation associated with that value of membership.

The dynamic analysis in Chapter 3 is based on an inherently different idea. It proposes a method by which fuzzy outcomes can be compared for the different stages of the problem. This composition maintains the compounded outcomes as fuzzy entities. The criterion for determining the "best" decision sequence is defined on the overall outcome.

The third situation discussed in this dissertation is the two-person decision problem. Fuzzy set-theoretic approaches to two-person games include models by Orlovosky [69], Ragade [76], and Butnariu [20]. Orlovosky describes a situation where the sets of feasible strategies of each player are defined as fuzzy sets. The model assumes that the outcomes are non-fuzzy. Ragade defines a fuzzy concept of rationality and in turn develops a notion of equilibrium. The model by Butnariu determines a fuzzy set of feasible strategies which is based on composing a player's outcome with his estimation of his opponent's behavior.

The analysis in Chapter 4, dealing with a two-person decision problem, assumes that the set of strategies is defined in a crisp way. The outcomes for each decision maker of possible combinations of strategies are expressed as fuzzy sets. The model differs from existing models in its definition of concepts such as equilibrium and solution. This difference is based on the distinction between the employed methods for choosing among fuzzy alternatives.

The final situation discussed in this dissertation is the multiple level or hierarchical decision problem. Fuzzy set-theoretic representations of such problems are not found in the literature. The model introduced in Chapter 5 draws on the basic principal/agent idea [78] for a two-level problem. This was previously formulated as an optimization problem where the decision makers at both levels are assumed to be utility maximizers. The upper-level (principal) maximizes its utility subject to the constraint that the utility of the lower-level (agent) is maximized. The analysis in Chapter 5 gives a fuzzy set-theoretic representation of this basic idea.

1.4 Example

In this section, a real-life decision problem is outlined using the systems representation adopted in this dissertation. The decision problem is described by Zebdeh [108] as an application to the use of fuzzy set theory in decision modeling. He develops a model for investigating cost variations for a manufacturing company in Virginia.

The company manufactures heavy equipment and consists of six departments (bonding, auto-line, press, finish, flange, and plating). Cost variations in each department are observed on a monthly basis. The question is whether to investigate, not to investigate, or to conduct a partial investigation into the incurred cost variation. These three alternatives form the set of possible decisions.

A simple solution to this decision problem would be to define ranges of variations in the cost for which a course of action is recommended. Zebdeh argues that such a solution could not be determined. This is because the outcome (cost/benefit analysis) associated with a decision is not well defined. This outcome is expressed to depend on the state of the system prior to the decision, the decision, and the state of the system after the decision.

Three possible states of the production system for each department are defined: in control; more or less out of control; and out of control. The decision maker is given the observed variation in cost for that period. He is then asked to define the current state of the system as a fuzzy set on the set of possible states.

Zebdeh suggests that this be done in two steps. Firstly, the decision maker is asked to represent the relationship between the state of the system (e.g. in control) and the variation in cost in a fuzzy way. That is, the decision maker defines each of the possible states as a fuzzy set on the universe of cost variations. Secondly, the observed cost variation is used to define the current state of the system as a fuzzy set whose elements are the possible states. For example, the membership of the state "in control" in the fuzzy set of "current state" is derived from the membership of the observed variation in the fuzzy set "in control."

The imprecision in determining the current state of the system cannot be attributed to a random occurrence since the variation has already been observed. The only source of imprecision is the subjective nature of assessing the current state of the system.

The decision maker is then asked to define the system state transitions for all possible states and decisions. For example, he is asked to determine, if the system is in control and the decision is not to investigate, what will the system state for the next period be. Fuzzy state transitions were determined. In other words, the future state is defined as a fuzzy set whose elements are the possible states.

Finally, the decision maker is asked to define the outcome for all possible states, decisions, and corresponding state transitions. The outcomes are defined as fuzzy sets on the universe of dollars. Zebdeh suggests that this be done in two steps. Firstly, the outcomes are defined by linguistic values such as high and low.

Secondly, these values are defined as fuzzy sets on the universe of dollars.

Each of these outcomes is defined for a specific state and a specific state transition. Fuzziness in the current state and in the state transition, previously defined by the decision maker, are incorporated using fuzzy set-theoretic operations. The fuzzy outcomes are then ranked according to a method suggested by Jain [48].

The decisions given by the fuzzy set-theoretic model and those implemented by the actual decision makers, for all six departments and over a period of six months, are compared. The model is shown to be representative of the decision process.

This example demonstrates the applicability of the theory. In this specific example the fuzzy sets defined by the decision makers are discrete. In other words, only a finite number of membership values are determined. For other applications [104], a piecewise linear membership function is suggested to represent continuous fuzzy sets. Shapes that are commonly used include triangular and trapezoidal functions.

CHAPTER 2
SINGLE-STAGE DECISION PROCESS

2.1 The Model

In this chapter a fuzzy set-theoretic model for the single-stage decision process is developed. The model refers to the representation given in Figure 1.1.

The model includes two basic mechanisms:

1) The state transition mechanism, which transforms the system from its present state to its future state as a result of an executed decision.
2) The objective mechanism, which determines the outcome of a decision for a specified state transition.

The fuzzy set-theoretic representation of these two mechanisms is given below.

2.1.1 The State Transition Mechanism

The state transition mechanism employs information about the present state of the system and the decision to determine the future state of the system. It is assumed that the decision maker defines:

1) a system state space denoted by S whose elements are $s \varepsilon S$;
2) the present state as a fuzzy set on the state space S

$$\theta = \{ u_\theta(s) \ / \ s \in S \} \text{ ; and} \tag{2.1}$$

3) The set of possible decisions, denoted by D, whose elements are $d \in D$.

A deterministic state transition can be defined by the following map:

$$F : S \times D \rightarrow S \tag{2.2}$$

and

$$F(s, d) = t \qquad \text{for } s, t \in S \text{ and } d \in D.$$

Since Equation (2.1) defines the present state of the system θ as a fuzzy set, the future state can only be determined as a fuzzy set. Therefore the map can be expressed as follows:

$$F(\theta, d) = \tau \tag{2.3}$$

for θ, τ fuzzy sets on S and for a given $d \in D$, where

$$\tau = \{ u_\tau(t) \ / \ t \in S \}. \tag{2.4}$$

The method used for mapping fuzzy sets is based on an axiomatic treatment introduced by Dubois and Prade [31]. Several variations on this extension principle exist in the literature [31]. The method adopted here is motivated as follows.

Suppose that the deterministic map is one to one, so that, there is only one $s \varepsilon S$ and one $t \varepsilon S$ such that $F(s, d) = t$. According to the method adopted here, $u_{\tau}(t) = u_{\theta}(s)$. This represents the idea that the imprecision in the definition of the future state is a direct result of imprecision in the definition of the present state.

Now suppose that the mapping is not one to one and that s, s', and $t \varepsilon S$ exist such that

$$F(s, d) = F(s', d) = t.$$

According to the method adopted here

$$u_{\tau}(t) = \max [u_{\theta}(s), u_{\theta}(s')].$$

This represents the idea that t may be the future state if either s or s' is the present state. Therefore the membership of t in the fuzzy set of future state τ is assigned the maximum membership value of s and s' in θ.

Therefore, in general

$$u_{\tau}(t) = \sup_{\substack{s: \\ F(s,d)=t}} u_{\theta}(s).$$

Thus far a deterministic map with fuzzy arguments was considered. Now suppose that the definition of the state transition is in itself imprecise. This can be represented as a fuzzy map:

$$F : S \times D \rightarrow \{ \xi \}, \tag{2.5}$$

where $\{ \xi \}$ is a collection of fuzzy sets each defined on the space S; or,

$$F(s, d) = \xi. \tag{2.6}$$

Using equations (2.1) and (2.6) the extension principle gives

$$F(\theta, d) = \tau \quad \text{and} \tag{2.7}$$

$\tau = \{ u_\tau(t) \,/\, t \varepsilon S \}$ where

$$u_\tau(t) = \sup_{\substack{s: \\ F(s,d)=\xi}} \min \left[u_\theta(s), u_\xi(t)\right]$$

The extension principle used here is more general than that in Equation (2.4). The minimum is introduced to represent the effect of imprecise mapping. The deterministic map can be considered as a special case of the fuzzy map, where $u_\xi(t) = 1$ for some $t \varepsilon S$ and is zero otherwise.

2.1.2 The Objective Mechanism

The objective mechanism determines the outcome of a decision for a given present state and corresponding state transition. Let the universe of the outcomes be denoted by B, and its elements by $b \varepsilon B$. Similar to the state transition, the objective can be defined as a map G. This map can be deterministic or fuzzy.

The deterministic map is

$$G : S \times D \times S \rightarrow B \qquad \text{and} \tag{2.8}$$

$$G(s, d, t) = b.$$

The fuzzy map is

$$G : S \times D \times S \rightarrow \{ \ell \}, \tag{2.9}$$

where $\{ \ell \}$ is a family of fuzzy sets each defined on the universe B.

The outcome of a decision is defined as a fuzzy set β on the universe B.

$$\beta = \{ u_\beta(b) \;/\; b \varepsilon B \}. \tag{2.10}$$

The decision problem is then that of ranking fuzzy outcomes of all possible decisions according to a predefined criterion.

The subject of ranking fuzzy outcomes defined over the same universe, according to a predefined criterion received much attention in the literature. The approaches can be grouped into two general classes. The first class deals with the determination of a real-valued function that maps each fuzzy set onto the real axis. This function provides an ordinal measure by which the fuzzy sets can be ranked. Some specific functions were defined by Yager [100], Chang [27], and Adamo [1].

The second class deals with the definition of an optimal set as a fuzzy set, whose elements are the possible outcomes. The membership grade represents a measure of "optimality" of the corresponding alternative. The alternatives are then ranked according to their membership grades. Some approaches in this class were considered by Baas and Kwakernaak [5], Baldwin and Guild [6], and Dubois and Prade [32].

Each of the above methods reflects an intuitive criterion of ordering. None, however, gives a conclusive prescription as may be applied to ranking of fuzzy outcomes. Admittedly, a unique criterion for ordering fuzzy outcomes may not exist. Indeed, the abundance of the suggested methods was a result of efforts to compensate for inadequacies of the prior methods [14].

Here, we limit this discussion to the ranking of fuzzy outcomes defined over an ordered continuous universe. The criterion is that of "minimizing" (or "maximizing") the outcome. New methods for determining the "best" decision by ranking such outcomes are suggested below. Further insight into these methods is given by examining their properties.

2.2. The Determination of the "Best" Decision

The outcome of a single-stage process depends on the present state, decision and future state of the system. The decision, however, determines the future state for a given present state. Therefore, for a given present state the outcome can be shown to depend only on the decision.

Assume that there are n possible decisions. Let the outcome of i-th decision be denoted by β_i. This outcome is defined as a fuzzy set on an ordered continuous universe B:

$$\beta_i = \{\ u_{\beta_i}(b_i)\ /\ b_i \varepsilon B\ \}. \tag{2.11}$$

Let the decision criterion be to minimize the outcome. The "minimum" outcome β^* is defined as follows:

$$\beta^* = \underset{i=1,\dots,n}{\widehat{\min}}\ [\beta_i], \tag{2.12}$$

where the "minimum" of the fuzzy sets is defined by the extension principle. Here, β^* is also a fuzzy set on B:

$$\beta^* = \{\ u_{\beta^*}(b^*)\ /\ b^* \varepsilon B\ \} \qquad \text{where} \tag{2.13}$$

$$u_{\beta^*}(b^*) = \sup_{\substack{(b_1,\dots b_n) \\ b^*=\min[b_i]}} \ \min_{i=1,\dots n}\ [u_{\beta_i}(b_i)].$$

Equation (2.13) gives the relationship between the membership function of the "minimum" outcome to those of all possible outcomes. The extension principle is used here in a more general form than that given in equation (2.4). In the earlier case, the relationship included only one fuzzy entity. Here, β^* depends on β_i for all $i=1,\dots,n$. This multiple dependency is treated by the minimum part of the

relationship. This states that for b^* to belong in β^* there should be $b_1,\dots,b_n$ belonging to $\beta_1, \dots, \beta_n$ such that $b^* = \min [b_1, \dots, b_n]$. The membership grade of b^* in the "minimum" outcome β^* is hence
the smallest of $u_{\beta_i}(b_i)$ for $i=1,\dots n$.

Therefore $u_{\beta^*}(b^*)$ is found using the part of equation (2.13) defined by the minimum operator. The supremum in that equation is included to cover the case of a many-to-one mapping, as discussed for equation (2.4).

Due to the nature of the extension principle the "minimum" outcome β^* is not necessarily identical to the outcome of any of the decisions. In other words, β^* may be a different fuzzy set from any of the β_i, $i=1,\dots,n$. Such a case is demonstrated in Figure 2.1. As a result, it may not be clear which decision is the best decision, or the decision that minimizes the outcome. More than one decision may yield an outcome that can be considered the minimum.

Let a level in [0, 1] be the level below which a decision may be considered the best decision. Then the "best" decision can be defined as the following fuzzy set:

$$\delta = \{ u_\delta(d) \; / \; d \varepsilon D \}. \qquad (2.14)$$

To determine the "best" decision, the membership function $u_\delta(d)$ has to be defined. This is done by determining the level below which the outcome of each decision is said to

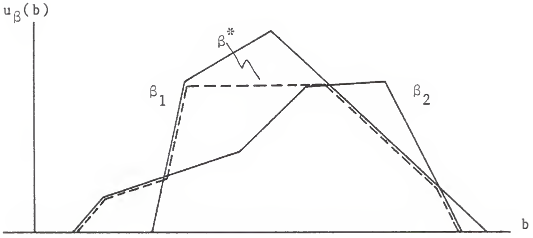
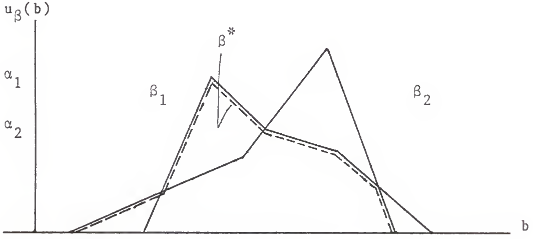


Figure 2.1 The "minimum" outcome



$\delta = [\ \alpha_1 / d_1,\ \alpha_2 / d_2\ ]$

Figure 2.2 "Best" decision in the weak sense match the "minimum" outcome. It is defined as the level below which a decision may be considered the "best."

Two types of methods are described for determining the fuzzy set of "best" decision. Each considers a different definition of matching the outcomes of decisions, β_i's, with the "minimum" outcome, β^*.

2.2.1 Determining the "Best" Decision in the Weak Sense: Method 1

Definition: Two fuzzy sets are defined to match in the weak sense at level α if the intersection of their α-cuts [31] is non-empty.

Proposition 2.1

If

$$\beta^* = \widetilde{\min}_{i=1,\ldots,n} [\beta_i]$$

and β_i's are nonempty, then there exists at least one decision such that

$$[\beta_i]_\alpha \cap [\beta^*]_\alpha \neq \phi \quad \text{for some } \alpha > 0.$$

where $[\cdot]_\alpha$ denotes the α-cut of a fuzzy set. The proof of this proposition follows from the definition of $\widetilde{\min}$ operator. It suggests that at least one decision may be considered the best at some α.

The next step is to determine the membership function of the fuzzy set of "best" decision. This is done in a way that reflects the match between the outcome of each decision and the "minimum" outcome.

Proposition 2.2

If two fuzzy sets match in the weak sense at level α, they also match at $\alpha' \leq \alpha$.

Proof

$[\beta_i]_\alpha \subset [\beta_i]_{\alpha'}$ for $\alpha' \leq \alpha$.

(from definition of α-cut)

Similarly

$[\beta^*]_\alpha \subset [\beta^*]_{\alpha'}$.

Therefore,

if $[\beta_j]_\alpha \cap [\beta^*]_\alpha \neq \phi$

then $[\beta_i]_{\alpha'} \cap [\beta^*]_{\alpha'} \neq \phi$. Q.E.D.

The membership grade of a decision in the fuzzy set of "best" decision, δ, is the level below which it can be considered to yield the "minimum" outcome. Hence, $u_\delta(d)$ is assigned the largest α for which its outcome matches the "minimum" outcome in the weak sense. Propositions 2.1 and 2.2 show that δ is not empty. That is, there is at least one decision for which $u_\delta(d) > 0$. The determination of the "best" decision in the weak sense as a fuzzy set is illustrated in Figure 2.2, and is summarized in the following procedure.

Procedure 2.1

```
- Find the "minimum" outcome $\beta^*$
- For i = 1, ..., n
    - Find the largest $\alpha$ such that
        $[\beta_i]_\alpha \cap [\beta^*]_\alpha \neq \phi$
    - Add decision-i to the fuzzy set of "best"
        decision, $\delta$, with a membership grade $\alpha$,
```

The method for determining the "best" decision in the weak sense does not impose any restrictions on the β_i's except that they are nonempty. It, therefore, can be considered as a general approach. It is considered a weak approach because the outcome of a decision is said to match the "minimum" outcome at some level α if the corresponding fuzzy sets overlap. The extent of this overlap is not taken into account. Requiring the equality of the corresponding α-cuts is a stronger sense of matching fuzzy sets. This is discussed in the next section.

2.2.2 Determining the "Best" Decision in the Strong Sense: Method 2

Definition: Two fuzzy sets are said to match in the strong sense at level α if their α-cuts are the same sets.

Theorem 2.1

Let β_1, β_2, ..., β_n be n distinct fuzzy sets defined over the same ordered continuous universe B. Let their membership functions be concave continuous functions of $b \varepsilon B$ for all i=1, ..., n. Also assume that the support$[\beta_i]$ is a bounded set for i=1, ..., n, where support$[\beta] \equiv \{ b \varepsilon B: u_\beta(b) > 0 \}$.

Let $\beta^* = \widetilde{\min}_{j=1,\ldots,n} \beta_j$.

If $[\beta_i]_\alpha \not\subset [\beta_j]_\alpha$ for all i, $j = 1,\ldots,n$ and $i \neq j$, for some $\alpha > 0$,

then there exists a unique decision i such that

$[\beta_i]_\alpha = [\beta^*]_\alpha$.

Proof

(Existence)

Let $b_0 = \inf\ [\beta^*]_\alpha$ for some $\alpha > 0$.

This implies that there exists a β_i such that

$b_0 = \inf\ [\beta_i]_\alpha$ and $b_0 \leq \inf\ [\beta_j]_\alpha$ for all $j=1,\ldots,n$.
(from definition of $\widetilde{\min}$.)

Hence $\inf\ [\beta_i]_\alpha \leq \inf\ [\beta_j]_\alpha$ for all $j=1,\ldots,n$.

Also $[\beta_i]_\alpha \not\subset [\beta_j]_\alpha$ for all $j=1,\ldots,n$, $j \neq i$.

(By assumption.)

Therefore $\sup\,[\beta_i]_\alpha \leq \sup\,[\beta_j]_\alpha$ for all $j=1,\ldots,n$.

This also implies that $\sup\,[\beta_i]_\alpha = \sup\,[\beta^*]_\alpha$.

(from definition of min.)

Since $u_{\beta_j}(b)$ is a continuous concave function for all $j = 1,\ldots,n$ (by assumption), then it can be concluded that

$[\beta_i]_\alpha = [\beta^*]_\alpha$.

(Uniqueness)

Let $[\beta^*]_\alpha = [\beta_i]_\alpha = [\beta_j]_\alpha$ for $i \neq j$.

This implies that $[\beta_i]_\alpha \subset [\beta_j]_\alpha$, which contradicts the original assumption. Therefore there is one and only one β_i such that

$[\beta_i]_\alpha = [\beta^*]_\alpha$ for some $\alpha > 0$. Q.E.D.

The conditions for which two fuzzy sets match in the strong sense at some level $\alpha>0$ are specified in Theorem 2.1 and are summarized below.

1) The membership functions of the outcomes of decisions, β_i's, are continuous and concave. This suggests that the membership function is maximum at some value on the

ordered universe of outcomes. Also a deviation from that value does not increase the corresponding membership.

2) At level α, the α-cut of the outcome of one decision does not include the α-cut of any other. This suggests that the outcomes of distinct decisions are essentially distinct although some overlap may occur.

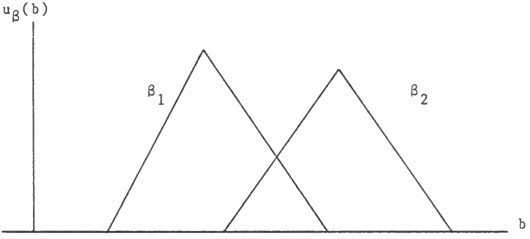
The next step is to determine the membership function of the fuzzy set "best" decision in accordance with the "strong" definition of matching fuzzy sets. Here, however, if two fuzzy sets match in the strong sense at level α, they do not necessarily match in the strong sense at level $\alpha' \leq \alpha$. This property corresponds to a special case defined by an additional assumption.

Figures 2.3.a. and 2.4.a. show two cases that will be treated seperately. In both cases the conditions specified by Theorem 2.1 hold. The first case also has the additional property stated in the following assumption.

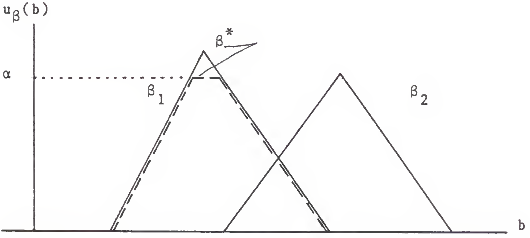
Case 1: special case

Assumption: Condition (2) holds for all $\alpha' \leq \alpha$

that is $[\beta_i]_{\alpha'} \not\subset [\beta_j]_{\alpha'}$ for all i,j
and $i \neq j$
and for all $\alpha' \leq \alpha$.



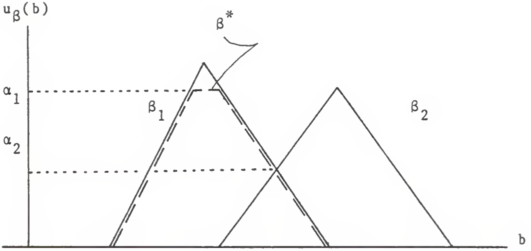
a) Case 1



$\delta' = [\ \alpha \ / \ d_1 \ ]$

b) "Best" decision in the strong sense

Figure 2.3 Comparison of "best" decisions: case 1

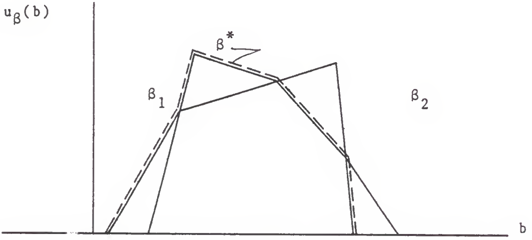


$\delta = [\ \alpha_1\ /\ d_1,\ \alpha_2\ /\ d_2\ ]$

$\delta' \subset \delta$

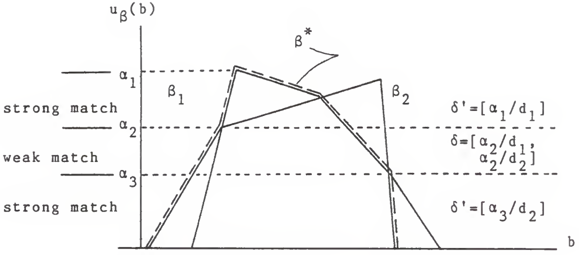
c) "Best" decision in the weak sense

Figure 2.3--continued



a) Case 2

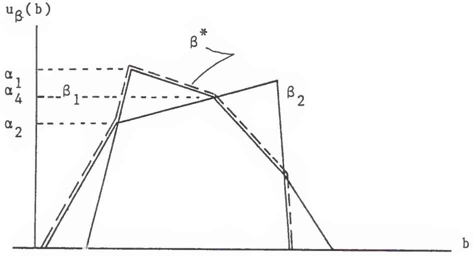
Figure 2.4 Comparison of "best" decisions: case 2



$$\delta'' = [\alpha_1/d_1] \cup [\alpha_2/d_1, \alpha_2/d_2] \cup [\alpha_3/d_2]$$

$$= [\alpha_1/d_1, \alpha_2/d_2]$$

b) "Best" decision: combination of strong/ weak sense.



$$\delta = [\ \alpha_1\ /\ d_1,\ \alpha_4\ /\ d_2\ ]$$

$$\delta' \sqsubset \delta$$

c) "Best" decision in the weak sense

Figure 2.4--continued

From Theorem 2.1

$[\beta_i]_{\alpha'} = [\beta^*]_{\alpha'}$ for one and only one i and for all $\alpha' \leq \alpha$.

This is demonstrated in Figure 2.3.a.

Let the fuzzy set "best" decision in the strong sense be denoted by δ'. In this case, the fuzzy set of "best" decision contains only one element (i-th decision) with a non-zero membership. Its membership grade is the level below which i-th decision is considered the "best" decision in the strong sense. It is assigned the largest value of α for which

$[\beta_i]_\alpha = [\beta^*]_\alpha$.

The procedure for determining the "best" decision for this special case is given in Figure 2.3.b. It is also summarized in the following.

Procedure 2.2

- Find the "minimum" outcome β^*
- Find the unique decision-i such that
  $[\beta_i]_\alpha = [\beta^*]_\alpha$
- The membership grade of decision-i in the fuzzy set of "best" decision, δ', is the largest α such that $[\beta_i]_\alpha = [\beta^*]_\alpha$.

The more general method given in Procedure 2.1 can also be used for this special case, as shown in Figure 2.3.c. It is easy to see that the "best" decision, δ', determined in the strong sense (by Procedure 2.2) is a subset of the "best" decision, δ, determined in the weak sense (by Procedure 2.1):

that is, $u_{\delta'}(d) \leq u_{\delta}(d)$ for all $d \varepsilon D$.

This is because

$[\beta_i]_{\alpha'} = [\beta^*]_{\alpha'}$ for all $\alpha' \leq \alpha$

implies that

$[\beta_i]_{\alpha'} \cap [\beta^*]_{\alpha'} \neq \phi$ for all $\alpha' \leq \alpha$.

Case 2: hybrid case

Consider the case given in Figure 2.4.a. Here the conditions specified by Theorem 2.1 hold, but not at all α levels. Therefore matching the "minimum" outcome with that of any of the decisions in the strong sense cannot be imposed at all levels. The method suggested here identifies the values (or ranges) of α for which the membership functions of β_i's conform to the conditions of the theorem from those that do not. The idea is to impose matching in the strong sense whenever possible.

Proposition 2.3

If two fuzzy sets match in the strong sense at level α, then they match in the weak sense for all $\alpha' \leq \alpha$.

That is, $[\beta_i]_\alpha = [\beta^*]_\alpha$

implies $[\beta_i]_{\alpha'} \cap [\beta^*]_{\alpha'} \neq \phi$ for all $\alpha' \leq \alpha$.

Proof

$[\beta_i]_\alpha = [\beta^*]_\alpha$

implies $[\beta_i]_\alpha \cap [\beta^*]_\alpha \neq \phi$.

Proof follows from Proposition 2.2.

This proposition suggests that procedures 2.1 and 2.2 can be combined to define another method that determines the "best" decision for this case.

Procedure 2.3

- Find the "minimum" outcome β^*
- Find ranges of α for which conditions of Theorem 2.1 hold
- Apply Procedure 2.2 in these ranges
- Apply Procedure 2.1 otherwise
- The fuzzy set of "best" decision, δ'', is the union of the fuzzy sets determined in Procedures 2.1 and 2.2.

This procedure is illustrated in Figure 2.4.b.

The fuzzy set of "best" decision is denoted here by δ''. It is defined as the union of the sets obtained from applying Procedures 2.1 and 2.2 in the relevant ranges. The membership grade of a decision in the fuzzy set, δ'', represents the level below which it can be considered the best. Here the term "best" is used as a combination of its strong and weak senses.

The general method given in procedure 1 can also be used to determine the "best" decision, δ, in the weak sense. This is given in Figure 2.4.c. It can be shown, as a direct result of Proposition 2.3, that δ includes δ''. That is

$u_{\delta''}(d) \leq u_{\delta}(d)$ for all $d \varepsilon D$.

The above methods give the solution to the decision problem as a fuzzy set of "best" decision defined on set D as opposed to a single decision. Every decision is associated with a level below which a decision is considered the best.

In the sequel, determining the "best" decision will follow the weak-sense definition given in Section 2.2. This is because the method is the most general among those developed in this chapter and does not impose restrictions on the fuzzy sets in question.

The determination of the "best" decision can be treated in a similar way when the decision criterion is that of maximizing the outcome. In this case the "maximum" outcome is defined as

$\beta^{*} = \underset{i}{\widetilde{\max}}\ \beta_{i}$ and similar analysis follows.

In the following chapters the basic notions of decision making with fuzzy information developed here will be utilized to model several more complicated problems. The next chapter will deal with multiple-stage decision making in the presence of imprecise information.

CHAPTER 3
MULTIPLE-STAGE DECISION PROCESS

3.1 The Model

The model of a multiple-stage decision process is developed in this chapter. The model refers to the diagram in Figure 1.2.

The number of stages of the decision model is assumed to be preset. Each stage of the process is modeled as given in Chapter 2. The composition of n such stages form the model of the multiple-stage decision process.

State Transition

The complete decision process consists of a dynamic sequence of decisions which in turn determines a sequence of state transitions. The initial state is the state of the system prior to the first decision. Also the final state is the state of the system after the n-th decision.

The initial state, θ, is defined as a fuzzy set on the state space S:

$$\theta = \{ u_\theta(s) / s \varepsilon S \}. \tag{3.1}$$

It is assumed that the state transition map is constant, but F can still be deterministic or fuzzy as described in Chapter 2.

Let $d_i \varepsilon D$ be the decision rendered at stage i. The state of the system after k stages is given by

$$F (F (F (\theta, d_1), d_2), \ldots, d_k). \tag{3.2}$$

To simplify the notation this is denoted by $F(\theta, d^k)$

$$\text{where} \quad d^k = (d_1, \ldots, d_k). \tag{3.3}$$

At every stage, k, $F(\theta, d^k)$ is a fuzzy set defined over S.

Objective

The outcome at every stage depends on the state of the system prior to the decision, the decision at that stage, and the state of the system after the decision.

Let β_k be the outcome at stage k.

$$\beta_k = G (F (\theta, d^{k-1}), d_k, F (\theta, d^k)). \tag{3.4}$$

In general, the outcome of each stage can be defined on a different universe. β_k is defined as a fuzzy set on the universe of outcome B_k for stage k.

The composition of the outcomes of each stage to that of the previous stages is represented in Figure 1.2 by block K.

Let C^k be the universe of composite outcomes of k stages. Then

$$K : B_k \times C^{k-1} \rightarrow C^k, \tag{3.5}$$

where $C^1 = B_1$.

That is, at every stage k the outcome is combined with the composite outcome of the (k-1) previous stages. The composite outcome of k stages is denoted by β^k; it depends on $\beta_1, \ldots, \beta_k$, and is defined as a fuzzy set on C^k.

In a similar fashion, a backward composition can be described. In that case

$$K : B_k \times C^{n-k+1} \rightarrow C^{n-k}, \tag{3.6}$$

where $C^n = B_n$.

The composite outcome of the last k stages is denoted by β^{n-k}, and depends on $\beta_k, \ldots, \beta_n$.

The map K can be, in general, deterministic or fuzzy. A more specific composition map is described in sequel.

The multiple-stage fuzzy set-theoretic model is an extension of the single-stage model. The determination of the "best" decision sequence, following the ideas described in Chapter 2, is given in the next section.

3.2 Determining the "Best" Decision Sequence

A sequence of n decisions is denoted by $d^n \varepsilon D^n$, where D^n denotes the n-th Cartesian power of set D. The "best" decision sequence is defined as a fuzzy set, δ^n on D^n;

$$\delta^n = \{u_{\delta^n} (d^n) / d^n \varepsilon D^n\}. \tag{3.7}$$

Here, two criteria for determining the "best" decision sequence are considered. The first criterion is that of achieving a preset target state for the system at the end of n stages. The second criterion is that of minimizing (or maximizing) the composite outcome defined on an ordered universe. It will also be shown that both criteria can be combined.

3.2.1 Achieving a Target State

In this section the sole criterion is that of achieving a preset target state. The outcomes of the decisions are not taken into consideration. It will be assumed that the target state can be defined as a fuzzy set, ψ, on the space S:

$$\psi = \{ u_{\psi}(s) / s \varepsilon S \}. \tag{3.8}$$

The state of the system at the end of n stages is given by $F(\theta, d^n)$. A decision sequence d^n is, hence, considered to lead to the target state if $F(\theta, d^n)$ matches ψ.

In Chapter 2 the matching of two fuzzy sets at level α was defined in the weak sense by

$$[\psi]_\alpha \cap [F(\theta, d^n)]_\alpha \neq \phi.$$

The fuzzy set of "best" decision is defined by its membership function. The membership value of d^n in the fuzzy set of "best" decision sequence, δ^n, is assigned the largest α value for which $F(\theta, d^n)$ matches ψ. It is the level below which d^n is considered to lead to the target state.

3.2.2 Minimizing the Composite Outcome

The criterion is that of minimizing the composite outcome of n stages. This is denoted by β^n. The map K determines β^n for all possible decision sequences. Given that β^n is defined as a fuzzy set on an ordered universe C^n, the "best" decision sequence can also be defined as a fuzzy set on D^n. This was described in Chapter 2.

Here, we take this a step further. Suppose that the "best" decision at stage k can be shown to depend only on stage k-1 (for forward composition) or on stage k+1 (for backward composition.) This suggests that a method similar to dynamic programming can be described for determining the "best" decision sequence.

A special case of the composition map K is considered. The supposition above is shown to hold for this composition.

Composition

Let the universe of outcomes be the same ordered universe at every stage:

$$B_1 = \dots = B_n = B;$$

and, let the composition be such that

$$K : B \times B \to B,$$

where $K(a, b) = a + b \qquad (a,\ b,\ a+b \in B)$

is a deterministic map defined by the sum.

Consider the backward composition. The composite outcome of the last two stages depends on the outcome of the individual stages. More specifically in this stage,

$$\beta^{n-1} = \beta_{n-1} \oplus \beta_n$$

where $\oplus$ denotes the sum of two fuzzy sets defined by the extension principle.

$$u_{\beta^{n-1}}(b) = \sup_{b=b'+b''} \min \left[u_{\beta_{n-1}}(b'),\ u_{\beta_n}(b'') \right].$$

The aim here is to describe a method for determining the "best" decision sequence in this case. The method utilizes backward (or forward) summation of fuzzy outcomes into fuzzy sets of composite outcomes. Suppose that β_{n-1} is the fuzzy outcome of stage n-1 for some decision. Also, suppose that β_n is the outcome of stage n of one decision, and β_n' is that of another. In order to be able to utilize the tools of dynamic programming it must be shown that

$$\beta_{n-1} \oplus \widetilde{\min} (\beta_n, \beta_n') =$$

$$\widetilde{\min} ((\beta_{n-1} \oplus \beta_n), (\beta_{n-1} \oplus \beta_n')).$$

This property will be shown true for all fuzzy sets defined on the same universe and whose sum and minimum are also fuzzy sets on the same universe.

Theorem 3.1:

Let X, Y, Z be fuzzy sets defined on the same ordered universe U. Then

$$X \oplus \widetilde{\min} (Y, Z) = \widetilde{\min} ((X \oplus Y), (X \oplus Z)).$$

(This is illustrated in Figures 3.1 and 3.2.)

Proof

Let x, y, $z \in U$ refer to elements in X, Y, Z respectively.

Let $\widetilde{\min}(Y, Z) \equiv P$

$X \oplus Y \equiv N$

$X \oplus Z \equiv M$

Let p, n, $m \in U$ refer to elements in P, N, M respectively.

Let $V \equiv X \oplus P$

$V' \equiv \widetilde{\min}(N, M)$

Let $v \in U$ refer to elements of V and V'.

To prove that $V = V'$, we need to show that

$u_V(v) = u_{V'}(v)$ for all $v \in U$.

From definition of $\widetilde{\min}$

$$u_P(p) = \sup_{p=\min(y,z)} \min(u_Y(y), u_Z(z))$$

From definition of $\oplus$

$$u_V(v) = \sup_{v=x+p} \min(u_X(x), u_P(p))$$

$$u_V(v) = \sup_{v=x+p} \min\left(u_X(x), \sup_{p=\min(y,z)} \min(u_Y(y), u_Z(z))\right)$$

$$u_V(v) = \sup_{v=x+p} \sup_{p=\min(y,z)} \min(u_X(x), u_Y(y), u_Z(z))$$

$$u_V(v) = \sup_{v=x+\min(y,z)} \min(u_X(x), u_Y(y), u_Z(z))$$

From definition of $\oplus$

$$u_N(n) = \sup_{n=x+y} \min(u_X(x), u_Y(y))$$

$$u_M(m) = \sup_{m=x+z} \min(u_X(x), u_Z(z))$$

From definition of $\widetilde{\min}$

$$u_{V'}(v) = \sup_{v=\min(n,m)} \min(u_N(n), u_M(m))$$

$$u_{V'}(v) = \sup_{v=\min(n,m)} \min\left(\sup_{n=x+y} \min(u_X(x), u_Y(y)), \sup_{m=x+z} \min(u_X(x), u_Z(z))\right)$$

$$u_{V'}(v) = \sup_{v=\min(n,m)} \sup_{\substack{n=x+y \\ m=x+z}} \min(u_X(x), u_Y(y), u_Z(z))$$

$$u_{V'}(v) = \sup_{v=\min((x+y),(x+z))} \min\,(u_X(x),\ u_Y(y),\ u_Z(z))$$

$$u_{V'}(v) = \sup_{v=x+\min(y,z)} \min\,(u_X(x),\ u_Y(y),\ u_Z(z))$$

$$u_{V'}(v) = u_V(v).$$

Q.E.D.

The theorem above suggests that comparisons of composite outcomes can be done sequentially. The method for determining the "best" sequence of decisions is summarized in procedure 3.1.

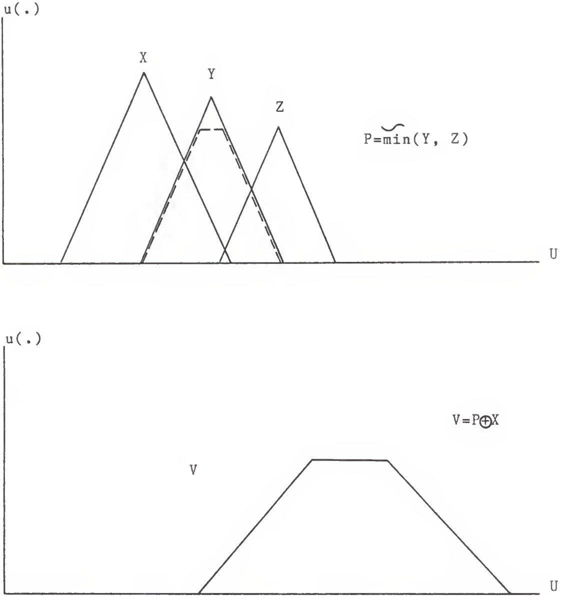


Figure 3.1 $V = X \oplus \widetilde{\min}\ (Y, Z)$

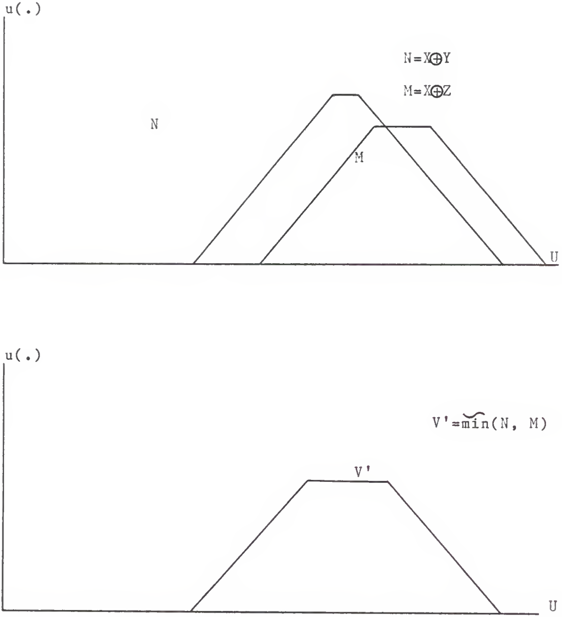


Figure 3.2 $V' = \widetilde{\min}\,((X \oplus Y), (X \oplus Z))$

Procedure 3.1

```
Define  $d^0 = [\phi]$

        $F(\theta, d^0) = \theta$

        $(\beta^0)^* = [\phi]$.

For every stage i from n-1 to 0
   for every $F(\theta, d^i)$, and
      for every $d \varepsilon D$:
           Find the outcome $\beta_i(d)$;
           Find the composite outcome of the last n-i stages

                $\beta^{n-i}(d) = (\beta_i(d) \oplus (\beta^{n-i-1})^*)$;

          Find the decision sequence of last n-i stages

                $d^{n-i} = (d, d^{n-i-1})$;

      find the "minimum" composite outcome of the last n-i
       stages

           $(\beta^{n-i})^* = \widetilde{\min}_{d \varepsilon D} \beta^{n-i}(d)$;

       find the "best" decision sequence of the last n-i
        stages (as in Chapter 2)

           $\delta^{n-i} = [ u_{\delta^{n-i}}(d^{n-i}) / d^{n-i} ]$.

Declare $\delta^n$ to be the "best" sequence of decisions.
```

This procedure is illustrated in Figure 3.3.

In this chapter multiple-stage decision processes were analyzed. Each stage of the process is modeled as a single unit decision maker, as developed in Chapter 2. The general principles of composing fuzzy outcomes and dynamic decision making can be applied to other forms of multiple-stage problems. In Chapter 4, two-person decision processes will be studied. The results of this chapter are applied to multiple-level decision processes in Chapter 5.

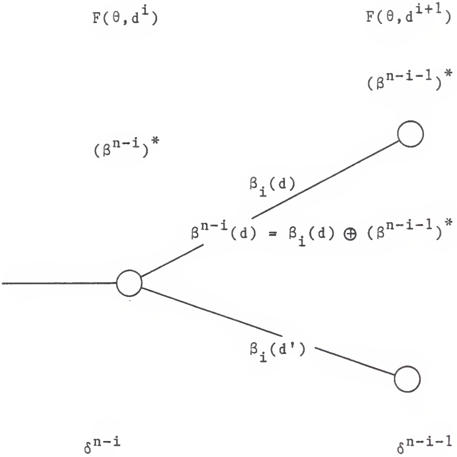


Figure 3.3 Determination of "best" decision sequence.

CHAPTER 4
TWO-PERSON DECISION PROCESS

4.1 The Model

In this chapter a two-person decision problem is modeled. The outcome of the process consists of two components. Each component represents the outcome of each decision maker. The situation is assumed to be that of conflict. That is, the component of the outcome corresponding to each decision maker depends not only on the his decision but also on that of his opponent.

In Figure 1.3 the estimation of the present state of the system, θ, is represented to be the same for both decision makers. Also the estimation of the state transition mechanism, F, is represented to be the same for both decision makers. These representations, however, are not binding. In general, each decision maker may have his own estimations. What is relevant to each decision maker is the set of decisions (strategies) of his opponent. It is according to his estimation that the effect of the opponent's strategy on his outcome is determined. Therefore, the sets of strategies, denoted by X and Y, are assumed to be known to both parties.

The objective block, G, consists of two sub-mechanisms corresponding to each of the decision makers. The outcome for the first decision maker of a pair of strategies $x \varepsilon X$ and $y \varepsilon Y$ is denoted by $\gamma_1(x, y)$. It is represented as a fuzzy set on an ordered universe:

$$\gamma_1(x, y) = \{ u_{\gamma_1(x, y)}(b) / b \varepsilon B_1 \}. \tag{4.1}$$

Similarly the outcome of the second decision maker is given by

$$\gamma_2(x, y) = \{ u_{\gamma_2(x, y)}(b) / b \varepsilon B_2 \}. \tag{4.2}$$

The outcomes for both decision makers can be represented as entries of a matrix. The rows correspond to the strategies of the first decision maker and the columns correspond to those of the second decision maker.

4.2 Equilibrium

Suppose that each decision maker wishes to maximize his outcome. Following the analysis given in Chapter 2, the "best" pair of strategies can be determined as a fuzzy set on X × Y. Here, however, a decision maker has no control over the actions of his opponent. An equilibrium represents a pair of strategies where no decision maker stands to gain from changing his strategy, given that his opponent does not change his.

4.2.1 The Determination of "Equilibrium"

In the fuzzy set-theoretic context, equilibrium is defined at level $\alpha = (\alpha_1, \alpha_2)$. A pair of strategies (x, y) is said to be in equilibrium at level α if the outcomes for decision makers 1 and 2 match the row and column "maximum" at levels α_1 and α_2 respectively. Here matching is defined, for the general case, in the weak sense. That is, (x, y) is in equilibrium at level α if

$$[\gamma_1(x, y)]_{\alpha_1} \cap [\Gamma_1(y)]_{\alpha_1} \neq \phi,$$

$$\text{where } \Gamma_1(y) = \underset{x \in X}{\widetilde{\max}}\, \gamma_1(x, y) \equiv \text{row "maximum,"}$$

and

$$[\gamma_2(x, y)]_{\alpha_2} \cap [\Gamma_2(x)]_{\alpha_2} \neq \phi,$$

$$\text{where } \Gamma_2(x) = \underset{y \in Y}{\widetilde{\max}}\, \gamma_2(x, y) \equiv \text{column "maximum."} \tag{4.3}$$

Using the above definition, an "equilibrium" pair of strategies, ζ, can be constructed as a fuzzy set on $X \times Y$:

$$\zeta = \{u_\zeta(x, y) \ / \ (x, y) \in X \times Y\}, \tag{4.4}$$

where $u_\zeta : X \times Y \rightarrow [0, 1] \times [0, 1]$

(this is known as type 2 fuzzy set [31]).

$u_\zeta(x, y)$ is assigned the level below which (x, y) is considered in equilibrium. This level has two components, each $\in [0, 1]$. The largest α_1 for which $\gamma_1(x, y)$ matches the row "maximum" and the largest α_2 for which $\gamma_2(x, y)$ matches the column "maximum," represent the components of this level.

The determination of the fuzzy set of "equilibrium" strategies, ζ, is summarized in the following procedure.

Procedure 4.1

For every $(x, y) \in X \times Y$:

$$\text{Find } \Gamma_1(y) = \underset{v \in X}{\widetilde{\max}} \ \gamma_1(v, y)$$

$$\text{and } \Gamma_2(x) = \underset{w \in Y}{\widetilde{\max}} \ \gamma_2(x, w);$$

Find the largest α_1 such that

$$[\gamma_1(x, y)]_{\alpha_1} \cap [\Gamma_1(y)]_{\alpha_1} \neq \phi$$

and the largest α_2 such that

$$[\gamma_2(x, y)]_{\alpha_2} \cap [\Gamma_2(x)]_{\alpha_2} \neq \phi;$$

Set $u_\zeta(x, y) = \alpha$, where $\alpha = (\alpha_1, \alpha_2)$.

4.2.2 The Existence of "Equilibrium"

The existence of equilibrium is represented by $\zeta \neq \phi$, that is, by the existence of (x, y) such that $u_\zeta(x, y) \neq 0$. This in turn suggests that the existence of "equilibrium" depends on the existence of (x, y) such that the equations in (4.3) hold for some $\alpha > 0$. The case where either α_1 or $\alpha_2 = 0$ converts the decision process to a one-person decision. It, therefore, corresponds to a trivial case of equilibrium. Here, the conditions under which the equations in (4.3) hold for $\alpha_1, \alpha_2 > 0$ are determined.

The following lemma and theorem are introduced to specify these conditions.

Lemma 4.1

Let $A_1, \ldots, A_n$ be n non-empty fuzzy sets on the same universe U, and $a_j \in U$ denote an element of A_j.

Also let $\Lambda = \widetilde{\max}_j A_j$.

Consider the i-th fuzzy set.

If $[A_i]_\alpha \cap [\Lambda]_\alpha \neq \phi$ for some $\alpha > 0$,

then there exists an $a_j \in [A_j]_\alpha$

such that $a_i \geq a_j$ for all j= 1, ..., n.

Proof

$[A_i]_\alpha \cap [\Lambda]_\alpha \neq \phi$

implies that their exists an $a_i \in U$, such that

$u_{A_i}(a_i) \geq \alpha$ and $u_\Lambda(a_i) \geq \alpha$.

From the definition of $\widetilde{\max}$

$$u_\Lambda(a_i) = \sup_{a_i = \max_j a_j} \min_j \left(u_{A_j}(a_j)\right)$$

This implies that there is at least one a_j in A_j such that

$a_i \geq a_j$ and

$u_{A_j}(a_j) \geq u_\Lambda(a_i) \geq \alpha$

Therefore there exists an $a_j \in [A_j]_\alpha$

such that $a_i \geq a_j$ for all j= 1, ..., n

Q.E.D.

Theorem 4.1

Suppose that $[A_j]_\alpha$ is a bounded set for all j = 1, ..., n.

$[A_i]_\alpha \cap [\Lambda]_\alpha \neq \phi$

if and only if

sup $[A_i]_\alpha \geq$ inf $[A_j]_\alpha$ for all j = 1, ..., n.

(First we prove the condition is necessary)

Given $[A_i]_\alpha \cap [\Lambda]_\alpha \neq \phi$.

Let $a_i \varepsilon [A_i]_\alpha$ and $a_i \varepsilon [\Lambda]_\alpha$.

Let sup $[A_i]_\alpha \geq a_i$.

From Lemma 4.1 there exists an $a_j \varepsilon [A_j]_\alpha$

such that $a_i \geq a_j$ for all j= 1, ..., n.

Also $a_j \geq$ inf $[A_j]_\alpha$.

Then

sup $[A_i]_\alpha \geq$ inf $[A_j]_\alpha$ for all j = 1, ..., n.

(Second we prove the condition is sufficient)

Given

$\sup\ [A_i]_\alpha \geq \inf\ [A_j]_\alpha$ for all $j = 1, \ldots, n$.

Let $a_i = \sup\ [A_i]_\alpha$

which emplies

$a_i \varepsilon\ [A_i]_\alpha$.

Also let $a_j = \inf\ [A_j]_\alpha$ for all $j = 1, \ldots, n$ $j \neq i$

which implies

$u_{A_j}(a_j) \geq \alpha$ for all $j = 1, \ldots, n$.

Then $a_i \geq a_j$ for all $j = 1, \ldots, n$

which implies $a_i = \max_j\ a_j$.

From the definition of $\widetilde{\max}$

$$u_\Lambda(a_i) = \sup_{a_i = \max_j a_j}\ \min_j\ (u_{A_j}(a_j))$$

$u_\Lambda(a_i) \geq u_{A_j}(a_j) \geq \alpha$.

Then, $a_i \varepsilon\ [\Lambda]_\alpha$.

Therefore

$[A_i]_\alpha \cap [\Lambda]_\alpha \neq \phi$ Q.E.D.

Assume that the supports of the fuzzy outcomes are bounded sets. That is

$[b \varepsilon B : u_{\gamma_1(x,y)}(b) > 0]$

and

$[b \varepsilon B : u_{\gamma_2(x,y)}(b) > 0]$

are bounded sets.

Then the result of Theorem 4.1 gives the necessary and sufficient conditions under which the equations in (4.3) are satisfied for $\alpha > 0$.

For a pair of strategies (x,y), the first equation in (4.3)

$$[\gamma_1(x, y)]_{\alpha_1} \cap [\Gamma_1(y)]_{\alpha_1} \neq \phi$$

is satisfied for some $\alpha_1 > 0$ if and only if

$$\sup [\gamma_1(x, y)]_{\alpha_1} \geq \inf [\gamma_1(v,y)]_{\alpha_1} \quad \text{for all } v \varepsilon X.$$

Similarly, the second equation

$$[\gamma_2(x, y)]_{\alpha_2} \cap [\Gamma_2(x)]_{\alpha_2} \neq \phi$$

is satisfied for some $\alpha_2 > 0$ if and only if

$$\sup [\gamma_2(x, y)]_{\alpha_2} \geq \inf [\gamma_2(x,w)]_{\alpha_2} \quad \text{for all } w \varepsilon Y.$$

Hence, under these conditions equilibrium is said to exist at α level. Also the fuzzy set of "equilibrium" pairs, ζ, is nonempty. These conditions can be interpreted to represent a "saddle point" at level α. The saddle point reflects the weakness in the matching requirement. This is because, at level α, the largest outcomes of conforming to the equilibrium are compared with the smallest outcomes of deviating from it.

Thus far, the concept of equilibrium has been examined. To complete the analysis of the two-person decision process a "solution" to the problem needs to be defined. In the next section this concept is introduced. It will also be shown that a solution is also an equilibrium according to the given definitions.

4.3 A Solution

A pair of strategies are considered a solution if they yield the "maximum" outcome for both decision makers. In the fuzzy set-theoretic context, a solution is defined at level $\alpha = (\alpha_1, \alpha_2)$. A pair of strategies (x, y) is said to be a solution at level α if the outcomes for decision makers 1 and 2 match the the "maximum" outcomes at levels α_1 and α_2, respectively. That is (x, y) is a solution at level α if

$$[\gamma_1(x, y)]_{\alpha_1} \cap [\Gamma_1]_{\alpha_1} \neq \phi$$

where $\Gamma_1 = \widetilde{\max}_{x,y \varepsilon X \times Y} \gamma_1(x, y)$

$\equiv$ "maximum" outcome for DM1
(Row and column maximum)

and

$$[\gamma_2(x, y)]_{\alpha_2} \cap [\Gamma_2]_{\alpha_2} \neq \phi$$

where $\Gamma_2 = \widetilde{\max}_{x,y \varepsilon X \times Y} \gamma_2(x, y)$

$\equiv$ "maximum" outcome for DM2
(Row and column maximum).

(4.5)

Using the above definition, a "solution" pair of strategies, σ, can be constructed as a fuzzy set on $X \times Y$:

$$\sigma = \{u_\sigma(x, y) \ / \ (x, y) \ \varepsilon \ X \times Y\} \tag{4.6}$$

where $u_\sigma : X \times Y \rightarrow [0, 1] \times [0, 1]$.

$u_\sigma(x, y)$ is assigned the level below which (x, y) is considered a solution. This level has two components, each in $[0, 1]$. The largest α_1 and the largest α_2 for which the equations in (4.5) are satisfied is the level, $u_\sigma(x,y)$.

The determination of the fuzzy set of "solution" strategies, σ, is summarized in the following procedure.

Procedure 4.2

Find $\Gamma_1 = \underset{x,y\varepsilon X\times Y}{\widetilde{\max}} \quad \gamma_1(x, y)$

and $\Gamma_2 = \underset{x,y\varepsilon X\times Y}{\widetilde{\max}} \quad \gamma_2(x, y)$;

For all $(x, y) \varepsilon X\times Y$;

Find the largest α_1 such that

$$[\gamma_1(x, y)]_{\alpha_1} \cap [\Gamma_1]_{\alpha_1} \neq \phi$$

and the largest α_2 such that

$$[\gamma_2(x, y)]_{\alpha_2} \cap [\Gamma_2]_{\alpha_2} \neq \phi;$$

Set $\alpha = (\alpha_1, \alpha_2)$ and $u_\sigma(x, y) = \alpha$

The existence of a solution can be treated in a similar manner as that of an equilibrium. The result of Theorem 4.1 gives the necessary and sufficient conditions under which the equations in (4.5) are satisfied for $\alpha > 0$. For a pair of strategies (x,y) the first equation in (4.5):

$$[\gamma_1(x, y)]_{\alpha_1} \cap [\Gamma_1]_{\alpha_1} \neq \phi$$

is satisfied for some $\alpha_1 > 0$ if and only if

$\sup\, [\gamma_1(x, y)]_{\alpha_1} \geq \inf\, [\gamma_1(v,w)]_{\alpha_1}$

for all $v,w \varepsilon X\times Y$.

Similarly, the second equation

$$[\gamma_2(x, y)]_{\alpha_2} \cap [\Gamma_2]_{\alpha_2} \neq \phi$$

is satisfied for some $\alpha_2 > 0$ if and only if

$\sup [\gamma_2(x, y)]_{\alpha_2} \geq \inf [\gamma_2(v,w)]_{\alpha_2}$

for all $v,w \in X \times Y$.

It can be easily shown, from the definitions, that a solution is also an equilibrium at level α.

This is because

$\Gamma_1 \subset \Gamma_1(y)$ for all $y \in Y$

and

$\Gamma_2 \subset \Gamma_2(x)$ for all $x \in X$.

Therefore $\sigma \subset \zeta$.

That is, a solution is also an equilibrium.

The basic two-person decision process developed here can be used to model more complex problems. In Chapter 5 one such application is shown in the context of multiple-level decision processes.

CHAPTER 5
MULTIPLE-LEVEL DECISION PROCESS

In this chapter a multiple-level decision process is considered. The multiple levels represent a hierarchical system where a decision is made at every level. These hierarchical decisions, starting from the top to the bottom level, cause the system to undergo a state transition. The representation of this hierarchy is given as a nested formulation of its different levels. A decision rendered at each level takes into account the decision levels that succeed it. For example, in a two-level decision process the upper level renders its decision first. In doing so, the upper level perceives the system to include the decision process of the successor level. The lower level, aware of the decision of the predecessor, in turn renders its own decision. The hierarchy can be extended to include other levels. This is achieved by defining the system at each level to include the decision processes of all the levels that succeed it.

The basic model of the two-level decision process with a single decision maker at each level is first developed. Section 5.4 deals with variations of this basic model.

5.1. The Model

Consider a two-level decision process. Let $x \varepsilon X$ be the decision of the upper level, and $y \varepsilon Y$ be the decision of the lower level. The sets of decisions are assumed to be known to the decision makers at both levels. It is also assumed that the upper level makes its decision x known to the lower level. Hence the decision of the lower level is conditioned on x, and will be denoted as y/x.

The outcome corresponding to the upper level depends both on x and y/x. The latter outcome is denoted by h(x, (y/x)) and is given as a fuzzy set on an ordered universe B_1. The outcome of the lower level depends on its decision y for a given upper-level decision x. The outcome is denoted by g(y/x), and is also given as a fuzzy set on an ordered universe B_2.

It is assumed that decision makers at both levels want to "maximize" their corresponding outcomes. Then the problem is that of determining the "best" decision for the upper level and the resulting "best" decision for the lower level. The upper level has control over the action of the lower level. This is because the outcome of the latter is conditioned on the decision of the former.

In the next section, it is assumed that the upper level knows the outcome of the lower level g(y/x) for all possible decisions. That is, the upper level can determine the "best" decision for the lower level for a given decision of its own. In Section 5.3 this assumption is relaxed. The outcome of the lower level is considered unknown by the upper level and can only be estimated.

5.2 Determining the "Best" Decision

The outcome of the upper level h(x, (y/x)) depends on the decision of lower level. Therefore determining the "best" decision for the upper level entails determining that for the lower level. For every $x \varepsilon X$ the "best" decision of the lower level is denoted by η/x, a fuzzy set on Y, and is determined by comparing $g(y/x)$ for possible $y \varepsilon Y$ and for a given x. The fuzzy set η/x is defined by its membership function. The membership function is obtained by the following procedure, based on the methods developed in Chapter 2.

Procedure 5.1

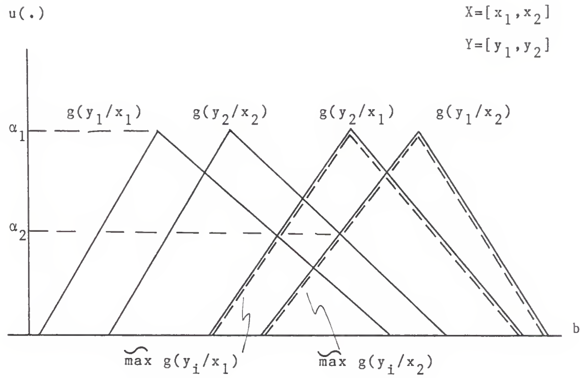
```
For every $x \varepsilon X$
    Determine $\underset{y \varepsilon Y}{\widetilde{\max}}\ g(y/x)$;
    For every $y \varepsilon Y$:
        Find the largest $\alpha$ such that
        $[g(y/x)]_\alpha \cap [\underset{y \varepsilon Y}{\widetilde{\max}}\ g(y/x)]_\alpha \neq \phi$.
```

$u_{\eta/x}(y) = \alpha$ where η/x is the fuzzy set of "best" decision of the lower level, for given upper level decision x.

The determination of the fuzzy set η/x is demonstrated in Figure 5.1.

Here the "best" decision of the lower level η/x is determined as a fuzzy set on Y. Therefore, using the extension principle, the outcome of the upper level can be expressed to depend on η/x instead of y/x. This is denoted by $h(x, (\eta/x))$, a fuzzy set on B_1. The determination of $h(x, (\eta/x))$ from $h(x, (y/x))$ and η/x is demonstrated in Figure 5.2.

The "best" decision of the upper level can now be determined, also as a fuzzy set on X. It is denoted by λ. The determination of λ is summarized in the following procedure and is demonstrated in Figure 5.3.



$\eta/x_1 = \{ \alpha_2|y_1, \alpha_1|y_2 \}$

$\eta/x_2 = \{ \alpha_1|y_1, \alpha_2|y_2 \}$

Figure 5.1 Determining the "best" decision for the lower level given the decision of the upper level

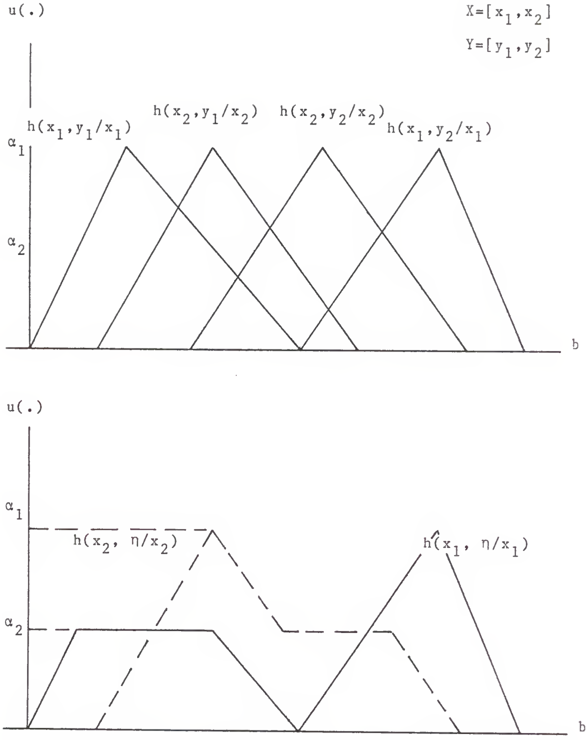
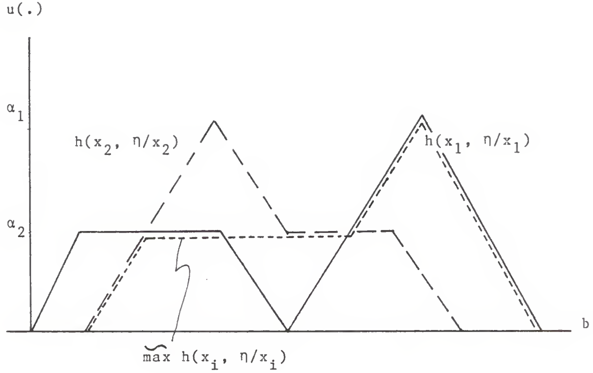


Figure 5.2 Determining $h(x_i, \eta/x_i)$ from $h(x_i,y_1/x_i)$, $h(x_i,y_2/x_i)$ and η/x_i.



$\lambda = (\alpha_1/x_1, \alpha_1/x_2)$

Figure 5.3 Determining the "best" decision for the upper level, λ.

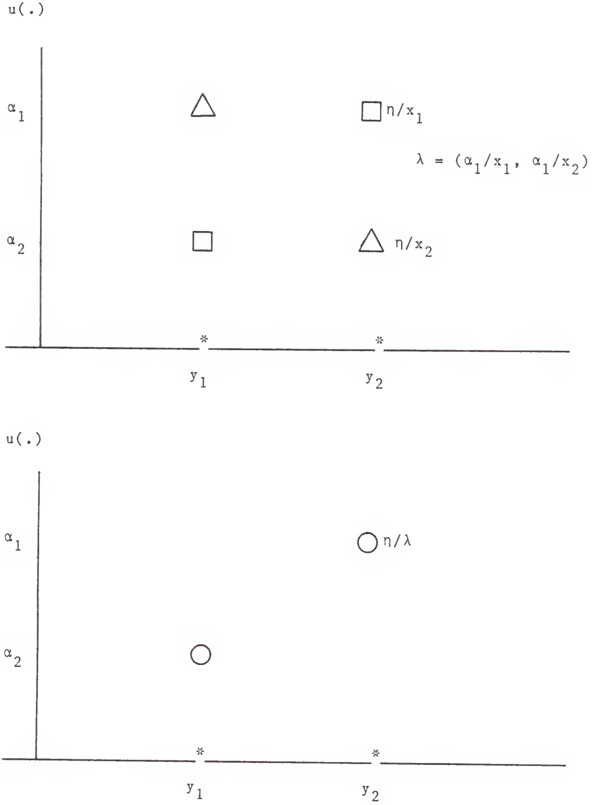


Figure 5.4 Determining η/λ from η/x_i and λ.

Procedure 5.2

Determine $\underset{x \varepsilon X}{\widetilde{\max}}\, h(x, (\eta/x))$.

For every $x \varepsilon X$:

Find the largest α such that

$$[h(x, \eta/x)]_{\alpha} \cap [\underset{x \varepsilon X}{\widetilde{\max}}\, h(x, \eta/x)]_{\alpha} \neq \phi,$$

$u_{\lambda}(x) = \alpha$, where λ is the fuzzy set of "best" decision of the upper level.

Thus far the "best" decision of the upper level has been determined as a fuzzy set λ on X. The "best" decision of the lower level η/x has been defined for a single x as a fuzzy set on Y. By combining this information, the "best" decision for the lower level can be expessed to depend on λ instead of x. This is denoted by η/λ and is obtained using the extension principle. The determination of η/λ from η/x for all x and from λ is demonstrated in Figure 5.4.

5.3 Usage of Estimates in the Multiple-Level Decision Process

Suppose that the decision maker at the upper level does not know the outcome of the lower level. In that case the former cannot control the decision of the latter. Nevertheless the outcome of the upper level still depends on the decision of the lower level.

Here, it is assumed that the upper level can estimate the outcome of the lower level for all possible decisions x and y. This estimate, denoted by g'(y/x), is expressed as a fuzzy set on B_2. Although the outcome of the lower level can only be estimated by the upper level, the universe on which it is defined is assumed to be known by both levels.

The estimatefor the outcome of the lower level can be used to carry out a similar analysis to that given in Section 5.2. Here, for a given upper-level decision x, an estimate of the "best" decision for the lower level can be determined. This utilizes procedure 5.1, where the outcome g(y/x) is substituted by its estimate g'(y/x). The estimated "best" decision for the lower level is conditioned on the decision of the upper level, is denoted by η'/x, and is defined as a fuzzy set on Y.

The best decision for the upper level, λ, can also be determined as a fuzzy set on x. This utilizes procedure 5.2 where η/x is substituted by η'/x.

Finally, the "best" decision for the lower level can be estimated for a given λ instead of x. This estimate is denoted by η'/λ and is a fuzzy set on Y. It is evaluated from η'/x and λ, as shown in the preceding section.

The estimate η'/λ may be a different fuzzy set from what the lower level considers its best decision η/λ. This is because the estimate determined by the upper level g'(y/x) may be different from what the lower level considers its outcome $g(y/x)$. In the following we will show that the relationship between η'/λ and η/λ depends on the relationship between $g'(y/x)$ and $g(y/x)$. This will help identify a class of estimates that can be utilized in multiple-level decision making.

Theorem 5.1

Let $p_1, \ldots, p_n$ be n fuzzy sets defined over a continuous ordered universe B. Let $q_1, \ldots, q_n$ also be n fuzzy sets defined over the same universe. Let $b_i \varepsilon B$ represent the elements of p_i and q_i.

If $p_i \subset q_i$ for $i=1, \ldots, n$,

then $\underset{i}{\widetilde{\max}}\, p_i \subset \underset{i}{\widetilde{\max}}\, q_i$.

Proof

Let $p = \underset{i}{\widetilde{\max}}\, p_i$ and $q = \underset{i}{\widetilde{\max}}\, q_i$

and $b \varepsilon B$ represent the elements of p and q.

Then

$$u_p(b) = \sup_{b=\max\{b_i\}} \min_i u_{p_i}(b_i)$$

and

$$u_q(b) = \sup_{b=\max\{b_i\}} \min_i u_{q_i}(b_i).$$

For i=1, ..., n, following the definition of fuzzy subsets we have

$$u_{p_i}(b_i) \leq u_{q_i}(b_i),$$

which implies

$$u_{p_i}(b_i) + \Delta_i(b_i) = u_{q_i}(b_i),$$

where $0 \leq \Delta_i(b_i)$ for $b_i \varepsilon B$ and i=1, ..., n.

Then

$$u_q(b) = \sup_{b=\max\{b_i\}} \min_i [u_{p_i}(b_i) + \Delta_i(b_i)].$$

Let $\Delta = \min_{\substack{b_i \varepsilon B \\ i=1,\ldots n}} \Delta_i(b_i).$

Then by substitution we have

$$u_q(b) \geq \sup_{b=\max\{b_i\}} \min_i [u_{p_i}(b_i)] + \Delta,$$

and

$$u_q(b) \geq \sup_{b=\max\{b_i\}} \min_i [u_{p_i}(b_i)].$$

Substituting the right-hand side by its definition we obtain $u_q(b) \geq u_p(b)$, which implies $p \subset q$. Q.E.D.

Theorem 5.2

Let p, q, ℓ and m be fuzzy sets defined over a continuous ordered universe B.

If $p \subset q$ and $\ell \subset m$. Then,

(i) $(p \cap \ell) \subset (q \cap m)$; and

(ii) $ht(p \cap \ell) \leq ht(q \cap m)$, where the height of a fuzzy set, ht(.), is the maximum membership to that set [31].

Proof

Part (i): Since $p \subset q$ and $\ell \subset m$, using the definition of fuzzy subsets we have

$u_p(b) \leq u_q(b)$ and $u_\ell(b) \leq u_m(b)$ for all $b \varepsilon B$.

From the definition of intersection

$u_{p \cap \ell}(b) = \min [u_p(b), u_\ell(b)]$

and

$u_{q \cap m}(b) = \min [u_q(b), u_m(b)]$.

Therefore

$u_{p \cap \ell}(b) \leq u_{q \cap m}(b)$ for all $b \varepsilon B$

and

$(p \cap \ell) \subset (q \cap m)$.

Part (ii): Since

$u_{p \cap \ell}(b) \leq u_{q \cap m}(b)$ for all $b \varepsilon B$

$\max_{b \varepsilon B} [u_{p \cap \ell}(b)] \leq \max_{b \varepsilon B} [u_{q \cap m}(b)]$

That is, $ht(p \cap \ell) \leq ht(q \cap m)$. Q.E.D.

Corollary 5.1

Let p_i, q_i, p and q be defined as in Theorem 5.1.

If $p_i \subset q_i$, then $(p_i \cap p) \subset (q_i \cap q)$ for $i=1,\ldots n$.

Proof

$p_i \subset q_i$ for $i=1,\ldots,n$ and Theorem 5.1

imply that $p \subset q$.

Therefore, using Theorem 5.2

$(p_i \cap p) \subset (q_i \cap q)$ Q.E.D.

In the context of multiple-level decision making, the results of the above Theorems and corollary can be interpreted as follows:

> If the outcome of the lower level, $g(y/x)$, is included in that estimated by the upper level, $g'(y/x)$, then the

"best" decision for the lower level, η/x, for a given upper decision, x, is also included in its estimate, η'/x.

That is, if $g(y/x) \subset g'(y/x)$ for all $x \varepsilon X$ and $y \varepsilon Y$, then $\eta/x \subset \eta'/x$ for all $x \varepsilon X$, where η/x and η'/x are fuzzy sets on Y. The various steps to this result are summarized as follows:

1) $g(y/x) \subset g'(y/x)$ implies $\underset{y}{\widetilde{\max}}\ g(y/x) \subset \underset{y}{\widetilde{\max}}\ g'(y/x)$ (from Theorem 5.1).

2) $\underset{y}{\widetilde{\max}}\ g(y/x) \subset \underset{y}{\widetilde{\max}}\ g'(y/x)$ implies

$$[g(y/x) \cap \underset{y}{\widetilde{\max}}\ g(y/x)] \subset [g'(y/x) \cap \underset{y}{\widetilde{\max}}\ g'(y/x)]$$

for all $y \varepsilon Y$ (from Theorem 5.2(i)).

3) (1) and (2) imply that

$$ht[g(y/x) \cap \underset{y}{\widetilde{\max}}\ g(y/x)] \leq ht[g'(y/x) \cap \underset{y}{\widetilde{\max}}\ g'(y/x)]$$

for all $y \varepsilon Y$ (from Theorem 5.2(ii)).

4) ht(.) is also the largest α for which the α-cuts of the indices have non-empty intersection. The largest α for which

$$[g(y/x)]_\alpha \cap [\underset{y}{\widetilde{\max}}\ g(y/x)]_\alpha \neq \phi$$

is the membership value of y in the fuzzy set η/x. Therefore $\eta/x \subset \eta'/x$.

This result can be taken a step further by considering that the "best" decision for the upper level, λ, is a fuzzy set on X.

Theorem 5.3

For λ a fuzzy set on X,

if $\eta/x \subset \eta'/x$ for all $x \in X$, then $\eta/\lambda \subset \eta'/\lambda$.

Proof

η/x, η'/x, η/λ and η'/λ are fuzzy sets on Y.
From the definition of η/λ,

$$u_{\eta/\lambda}(y) = \sup_{x} \min [u_{\eta/x}(y), u_{\lambda}(x)].$$

Similarly,

$$u_{\eta'/\lambda}(y) = \sup_{x} \min [u_{\eta'/x}(y), u_{\lambda}(x)].$$

Since $u_{\eta/x}(y) \leq u_{\eta'/x}(y)$, then

$$u_{\eta/\lambda}(y) \leq u_{\eta'/\lambda}(y)$$

Therefore,

$\eta/\lambda \subset \eta'/\lambda$ Q.E.D.

The overall result is therefore given by:

If $g(y/x) \subset g'(y/x)$ then $\eta/\lambda \subset \eta'/\lambda$.

This result describes the relationships of the estimated lower-level decision parameters used during the upper-level decision making. Specifically, it states that if the estimated outcome includes the outcome itself, then the decision inferred from that estimate includes the decision itself.

5.4 Extensions of the Basic Model

Thus far a two-level decision process has been considered in this chapter. The model drew upon the results of Chapter 2, since both levels were modeled to contain a single-stage decision problem.

In general, a multiple-level decision process may consist of more than two levels. Also, within each level the decision process itself may be a a more complex problem than that described in Chapter 2. In this section three variations of the basic model are described. It will be shown that the structure of the basic model can be maintained while describing more complex situations.

The first variation describes a two-level process with multiple stages. The second variation describes a two-level process with intra-level conflict. Finally, a third variation describes a three-level problem.

5.4.1 Two-Level Multiple Stage Problem

Consider a two-level decision process where for every upper-level decision the lower level renders a sequence of decisions. In this case the decision process of the lower level is a multiple-stage problem. For every upper-level decision, the "best" decision sequence for the lower level can be determined. This draws upon the results given in Chapter 3. This sequence of decisions can in turn be utilized to determine the "best" decision for the upper level as described in Section 5.2.

Suppose that the upper-level decision problem is also a multiple-stage problem. This means that the upper level needs to determine a decision sequence of its own. For every stage corresponding to an upper-level decision process, a multiple-stage decision problem for the lower level is considered. The dynamic two-level decision problem is therefore a nested formulation of multiple-stage problems. This process is represented in Figure 5.5.

5.4.2 Two-Level Decision Problem with Intra-level Conflict

Consider a two-level decision process where the lower level consists of two decision makers. This was depicted in Figure 1.4(b). In that case the lower-level process is a two-person decision problem. This problem was analyzed in Chapter 4. A pair of strategies were defined to constitute

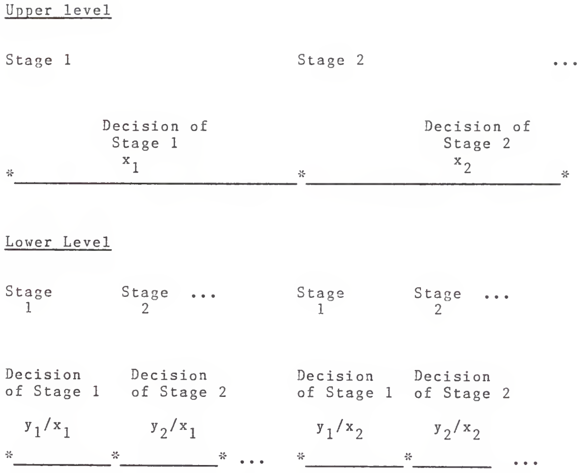


Figure 5.5 Two-level multiple-stage problem

a decision. The "best" decision of the lower level for a given upper-level decision is the solution to the intra-level problem. Consequently, this solution can be used to determine the "best" decision of the upper level, as shown in Section 5.2.

5.4.3 Three-Level Decision Problem

The three-level decision process contains the upper, middle and lower levels of decision, rendered in that order. The decision of the lower level is conditioned on the decisions of both the middle and upper levels. Also the decision of the middle level is conditioned on that of the upper level. It takes into account the possible decisions of the lower level. Finally, the decision of the upper level takes into account the decisions of both the middle and lower levels.

This process can be modeled as two nested two-level problems. That is, in the first problem, the lower level is itself a two-level problem. Determining the best decision for every level utilizes the results Section 5.2 recursively to both problems, as shown in Figure 5.6.

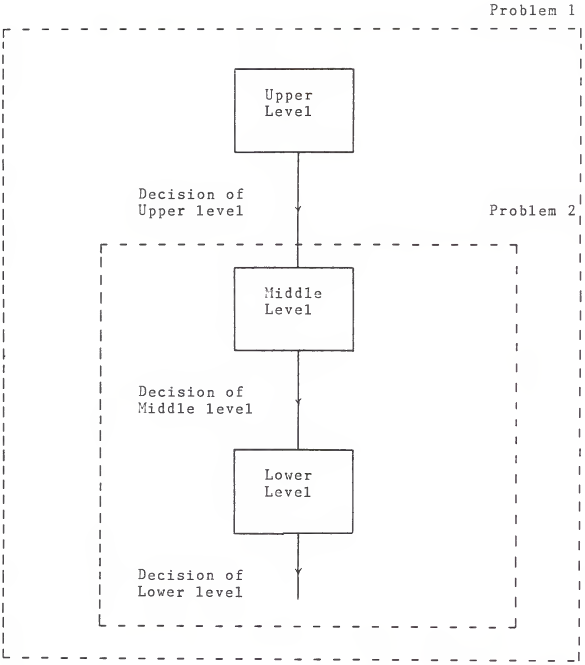


Figure 5.6 Three-level decision problem.

CHAPTER 6
CONCLUSION

This dissertation focuses on the development of fuzzy set theory as a mathematical tool for decision making in the presence of imprecise information. In many situations, modeling the imprecision in decision making using probabilistic approaches may not be possible. Determining objective probability distributions to describe imprecision in decision processes is often a difficult task and may involve a considerable amount of data collection. Moreover, some sources of imprecision may not be stochastic in nature. That is, they may not involve random occurrences. They may result from subjective assessments made by decision makers. In these cases fuzzy set-theoretic approaches give alternatives to modeling imprecision in decision making. The operations of fuzzy set theory involve simple mathematical relations, (maximum or minimum.) This is in acknowledgement of the limitations of dealing with imprecision outside the axiomatic treatment of probability theory. It is what many believe makes the theory representative of some real-life situations.

Fuzzy set theory has been successfully implemented in many applications [59]. Most of the more recent applications have been in the field of artificial intelligence [63] [71] [80]. The extent of its usage, however, is limited to simple functions, namely choosing among alternatives. This is because the theory, as a tool for decison making does not impose a unique structure to problem solving. The literature in the field contains a collection of ad hoc methods based on a variety of reasonings.

The general contribution of this dissertation is to utilize fuzzy set theory to develop a comprehensive tool to deal with complex decision-making situations. The aim is to take this research a step beyond developing yet another interpretation of fuzzy set theory as a tool for decision making. It provides consistent building blocks that can be used to construct decision models for various situations. Such situations include the single-stage decision problem (Chapter 2,) the multiple-stage decision problem (Chapter 3,) the two-person decision problem (Chapter 4,) and the multiple- level decision problem (Chapter 5.) The substance of this study revolves around the properties of these models. Parallellism between the fuzzy set-theoretic models and deterministic models is drawn to emphasize these properties.

The basic building block developed in this dissertation is a method for comparing fuzzy alternatives. The main difference between this method and the existing ones is that it neither dictates nor describes the behavior of decision makers. It simply propagates the fuzziness in the alternatives onto the choice of the "best" alternative. This acknowledges that, in this context, there may not exist a unique reasoning that supports the choice of one alternative over another.

A fuzzy set-theoretic representation of a dynamic decision model is introduced in this research (Chapter 3.) This model considers several stages of the decision process. The composition of these stages is described. The properties of this model led to the development of a technique parallel to dynamic programming. This technique is employed to determine the "best" sequence of decisions in a fuzzy sense.

A two-person conflict problem is also described in the fuzzy set theoretic context (Chapter 4.) This problem was previously discussed in the literature [20] [70] [76]. The basic difference of the model given in this dissertation from the previous approach lies in the method of comparing alternatives. In a manner parallel to a two-person game, concepts of equilibrium and solution in a fuzzy sense are introduced. In addition, the conditions for their existance are derived.

Finally, the multiple-level decision problem is modeled using fuzzy set theory (Chapter 5.) This problem involves a hierarchical decision process. A two-level problem is first introduced. Its model utilizes the basic principal-agent idea [78] to develop a fuzzy set-theoretic model. Possible variations on this basic model are outlined.

Future research in this field includes developing models for other types of decision processes, for example multi-objective and n-person decision processes. The immediate concern, however, would be to implement some of the theoretical models described in this dissertation.

APPENDIX
FUZZY SET DEFINITIONS AND OPERATIONS

Fuzzy Sets

Let X be a given set. We will call X the universe. A fuzzy set Ω in X is defined by a function:

$u_\Omega : X \to [0,1]$.

$u_\Omega(x)$ is called the membership grade of x in Ω
We adopt the following notation of Zadeh [102]:

$\Omega = \{u_\Omega(x) \;/\; x \varepsilon X\}$

The largest membership grade to a fuzzy set Ω is called the hight of a fuzzy set, and is denoted by ht(Ω).

If u_Ω assumes only the values 0 or 1 then Ω is

identified with the set $\{x \;\varepsilon\; X : u_\Omega(x) = 1\}$

In this case Ω is called a crisp set.

Subset of a fuzzy set

A fuzzy set Λ is a subset of another fuzzy set Ω, both defined on the same universe X, if and only if

$u_{\Lambda}(x) \leq u_{\Omega}(x) \qquad$ for all $x \in X$.

Intersection of fuzzy sets

The intersection of two fuzzy sets, Λ and Ω, defined on the same universe, X, is given by

$$u_{\Lambda\ \ \Omega}(x) = \min\ (u_{\Lambda}(x),\ u_{\Omega}(x))$$

for all $x \in X$

Union of fuzzy sets

The union of two fuzzy sets, Λ and Ω, defined on the same universe, X, is given by

$$u_{\Lambda\ \ \Omega}(x) = \max\ (u_{\Lambda}(x),\ u_{\Omega}(x))$$

for all $x \in X$

Extension principle

Let Ω1, Ω2, ..., Ωr be fuzzy sets in X1, X2, ..., Xr respectively, and let

$Q: X1 \times X2 \times \ldots \times Xr \rightarrow Y$ be a mapping.

Q defines a fuzzy set

$\Lambda = \{u_\Lambda(y) \ / \ y \varepsilon Y\}$

in Y. The membership grade of y in Λ is defined by

$$u_\Lambda(y) = \sup_{(x1, \ldots, xr) \varepsilon Q^{-1}(y)} \min [u_{\Omega 1}(x1), \ldots, u_{\Omega r}(xr)]$$

where $Q^{-1}(y)$ is the preimage of y:

$Q^{-1}(y) = \{ (x1, \ldots, xr) : Q(x1, \ldots, xr) = y \}$

Λ is called the Q-extension of $\Omega 1$, $\Omega 2$, ..., Ωr [31].

The notation adopted is as follows:

$\Lambda = Q(\Omega 1, \ldots, \Omega r)$

Two special cases of Q-extension are the minimum-extension (min), and the maximum-extension (max) [31]. They are defined for the case where X1= ...= Xr= X and X is an ordered universe.

The minimum-extension is defined by the minimum function.

$y = Q(x1, \ldots, xr) = \min (x1, \ldots, xr)$

$\Lambda = \min (\Omega 1, \ldots, \Omega r)$.

Similarly, the maximum-extension is defined by the maximum function.

$y = Q(x1, \ldots, xr) = \max (x1, \ldots, xr)$

and $\Lambda = \max (\Omega 1, \ldots, \Omega r)$.

α-cut

Let $0 \leq \alpha \leq 1$ be given. The α-cut [31] of a fuzzy set Ω in X is the crisp set

$\Omega_{\alpha} = \{x \in X : \alpha \leq u_{\Omega}(x)\}$.

Support

The support of a fuzzy set Ω defined on X is the crisp set

$\text{support } [\Omega] = \{x \in X : 0 < u_{\Omega}(x)\}$

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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